3.

Functions

- One-to-one and many-to-one
- Combining functions
- Using the output from one function as the input of another
- Domain and range of functions of the form $f \circ g(x)$
- Inverse functions
- Graphical relationship between a function and its inverse
- Condition for the inverse to exist as a function
- The absolute value function
- The absolute value of x as $\sqrt{x^2}$
- Solving equations involving absolute values
- Solving inequalities involving absolute values
- The graph of $y = \dfrac{1}{f(x)}$
- The graph of $y = \dfrac{f(x)}{g(x)}$ for $f(x)$ and $g(x)$ polynomials
- Absolute value – extension activity
- Rational functions – extension activity
- Miscellaneous exercise three

One-to-one and many-to-one

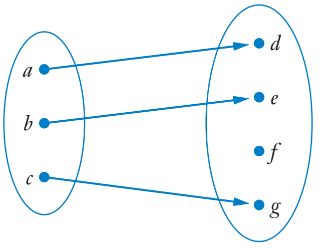
As the *Preliminary work* suggested, you should already be familiar with the idea that in mathematics, a rule that takes one element from one set and assigns to it one, *and only one*, element from a second set, is called a **function**. You should also be familiar with the associated terms **domain**, **natural domain** and **range**.

From your earlier studies you may also be familiar with the idea that a function may be **one-to-one**, or **many-to-one**. These terms, and a few others, are explained below.

In the function diagram shown on the right the **domain** is the set $\{a, b, c\}$.

We say that each element from the first set, the domain, **maps onto** an element of the second set, the **co-domain**, $\{d, e, f, g\}$. Those elements of the co-domain that the elements of the first set map onto, form the **range**, in this case $\{d, e, g\}$.

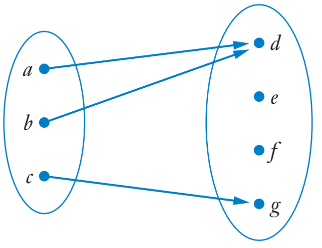
With each element of the domain mapped onto a different element of the range this function is said to be **one-to-one**.



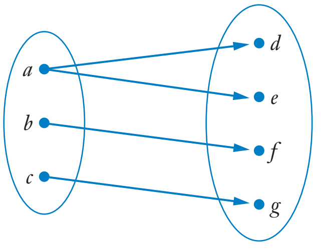
Contrast the above situation with that shown in the new diagram on the right.

Now more than one element of the domain map onto the same element of the range. In this case $a \to d$ and $b \to d$.

We call such functions **many-to-one**.



One-to-many relationships can occur, as shown on the right, but under the requirement that a function takes one element from the domain and assigns to it one, *and only one*, element of the range, a one-to-many relationship would *not* be called a function.



Note

Whilst the diagrams above show letters as the elements of the domain and range **the functions we will deal with in this chapter will have domains and ranges consisting only of real numbers**, i.e. numbers from $\mathbb{R}$, the set of real numbers.

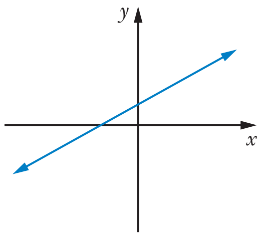
Functions whose values are real numbers, are sometimes referred to as real-valued functions.

ISBN 9780170395274

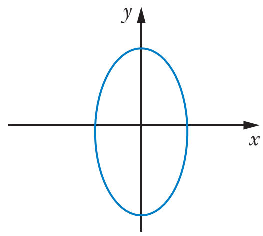
You should also be familiar with the fact that our requirement that a function takes one element from the domain and assigns to it one *and only one* element from the range, means that the graph of a function must be such that:

If a vertical line is moved from the left of the domain
to the right it must never cut the graph in more than one place.

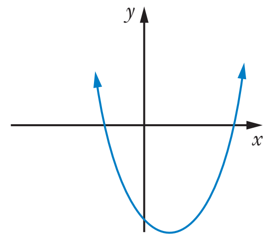
This is called the **vertical line test**.



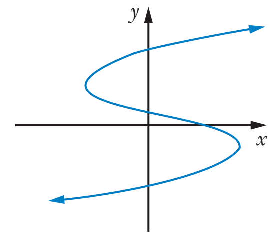
A function



Not a function

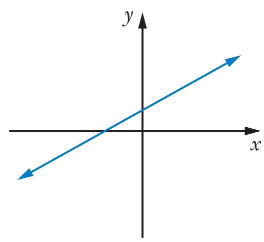


A function

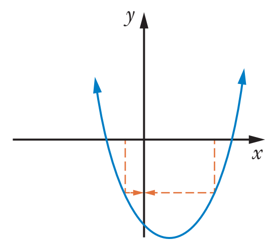


Not a function

We could use a similar **horizontal line test** to determine whether a function is a one-to-one function or not. Thus of the two graphs shown above that show functions, only the first would pass the horizontal line test.



A one-to-one function



A many-to-one function

To be a one-to-one function the graph needs to pass both the vertical line test (to be a function), and the horizontal line test (to be one-to-one).

Combining functions

We can use the basic operations of +, –, × and ÷ to combine functions.

For example, if $f(x) = x + 1$ and $g(x) = x - 1$ then

$$\begin{aligned} f(x) + g(x) &= (x+1) + (x-1) \\ &= 2x \end{aligned}$$

$$\begin{aligned} f(x) - g(x) &= (x+1) - (x-1) \\ &= 2 \end{aligned}$$

$$\begin{aligned} f(x) \times g(x) &= (x+1)(x-1) \\ &= x^2 - 1 \end{aligned}$$

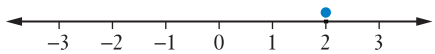
$$\begin{aligned} f(x) \div g(x) &= (x+1) \div (x-1) \\ &= \frac{x+1}{x-1} \end{aligned}$$

ISBN 9780170395274

However we do have to be careful when considering the domain and range of each of these new functions formed by combining $f(x)$ and $g(x)$. The functions $f(x)$ and $g(x)$ defined on the previous page each have domain $\mathbb{R}$ and range $\mathbb{R}$ but this does not mean that the functions formed by combining $f(x)$ and $g(x)$ will necessarily have domain $\mathbb{R}$ and range $\mathbb{R}$.

For example:

- $f(x) - g(x) = 2$ and therefore has range $\{y \in \mathbb{R}: y = 2\}$.

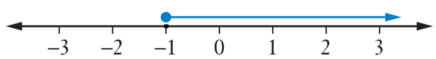


Remember

- Reading '∈' as 'is a member of', and ':' as 'such that', then $\{y \in \mathbb{R}: y = 2\}$ can be read as:

  y is a member of the set of real numbers such that y equals 2.
- We could use any letter to define the range and the domain but we will tend to use *x* when defining a domain and *y* when defining a range.

- $f(x) \times g(x) = x^2 - 1$ and therefore has range $\{y \in \mathbb{R}: y \geq -1\}$. i.e:

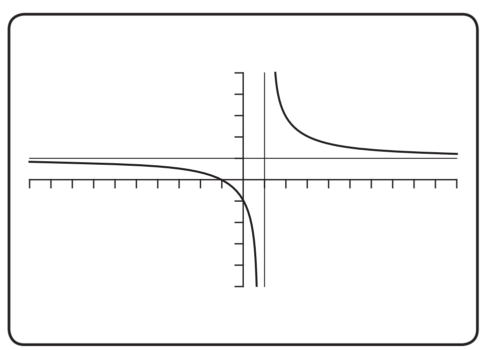


- $\dfrac{f(x)}{g(x)} = \dfrac{x+1}{x-1}$ is undefined for $x = 1$.

  Thus $\dfrac{f(x)}{g(x)}$ does not exist as a function unless we restrict the domain of $g(x)$ to $\{x \in \mathbb{R}: x \neq 1\}$.

  With this restriction in place $\dfrac{f(x)}{g(x)}$ does exist and has domain $\{x \in \mathbb{R}: x \neq 1\}$ and range $\{y \in \mathbb{R}: y \neq 1\}$.

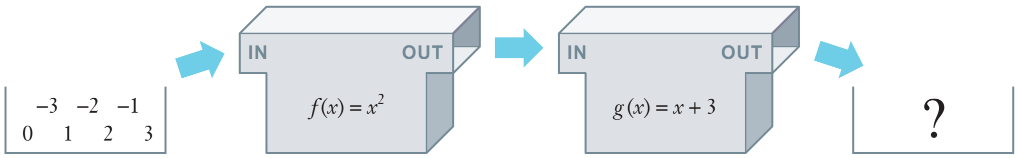
  The display of $y = \dfrac{x+1}{x-1}$ shown on the right suggests agreement with these facts. The function seems to have $x = 1$ as a vertical asymptote and $y = 1$ as a horizontal asymptote suggesting that the function can cope with x values close to 1, but not 1 itself, and it can output values close to 1, but not 1 itself.



ISBN 9780170395274

Using the output from one function as the input of another

The section in the *Preliminary work* on function mentioned the idea of using the output from one function as the input of a second function.



With the domain stated, combining the functions f and g in this way will give a final output of $\{3, 4, 7, 12\}$:

As the *Preliminary work* mentioned:

We write this combined function as $g[f(x)]$

or as $g \text{ o } f(x)$ or $g \circ f(x)$ for 'g of f of x'

or as $gf(x)$,

and, though our 'machine diagram' above shows the 'f function' first, we write the combined function as $gf(x)$, to show that the 'f function', being closest to the '(x)', operates on the x values first.

EXAMPLE 1

With $f(x) = 2x - 3$, $g(x) = (x + 1)^2$ and with an initial domain of $\{0, 1, 2, 3, 4\}$, determine the range of

a $gf(x)$,

b $fg(x)$,

c $gg(x)$.

Solution

a $\{0, 1, 2, 3, 4\} \xrightarrow{f(x)} \{-3, -1, 1, 3, 5\} \xrightarrow{g(x)} \{0, 4, 16, 36\}$

The range is $\{0, 4, 16, 36\}$.

b $\{0, 1, 2, 3, 4\} \xrightarrow{g(x)} \{1, 4, 9, 16, 25\} \xrightarrow{f(x)} \{-1, 5, 15, 29, 47\}$

The range is $\{-1, 5, 15, 29, 47\}$.

c $\{0, 1, 2, 3, 4\} \xrightarrow{g(x)} \{1, 4, 9, 16, 25\} \xrightarrow{g(x)} \{4, 25, 100, 289, 676\}$

The range is $\{4, 25, 100, 289, 676\}$.

ISBN 9780170395274

EXAMPLE 2

Given that $f(x) = 3x - 2$, $g(x) = 2x + 1$ and $h(x) = x^2$ express each of the following functions in a similar way (i.e. in terms of x).

a $g \circ f(x)$ **b** $f \circ f(x)$ **c** $f \circ h(x)$ **d** $h \circ f(x)$

Solution

a
$$\begin{aligned} g \circ f(x) &= g[f(x)] \\ &= g[3x - 2] \\ &= 2(3x - 2) + 1 \\ &= 6x - 3 \end{aligned}$$

b
$$\begin{aligned} f \circ f(x) &= f[f(x)] \\ &= f[3x - 2] \\ &= 3(3x - 2) - 2 \\ &= 9x - 8 \end{aligned}$$

c
$$\begin{aligned} f \circ h(x) &= f[h(x)] \\ &= f[x^2] \\ &= 3x^2 - 2 \end{aligned}$$

d
$$\begin{aligned} h \circ f(x) &= h[f(x)] \\ &= h[3x - 2] \\ &= (3x - 2)^2 \end{aligned}$$

Domain and range of functions of the form $f \circ g(x)$

If we are not told a specific domain for a composite function we assume it to be the **natural** or **implied** domain, i.e. all the real numbers for which the composite function is defined. However care needs to be taken because the natural domain of $f[g(x)]$ may not simply be the natural domain of $g(x)$.

EXAMPLE 3

State the natural domain and the corresponding range of each of the following functions given that $f(x) = x - 5$ and $g(x) = \dfrac{1}{x-1}$.

a $g \circ f(x)$, **b** $f \circ g(x)$.

Solution

a First identify the natural domain and range of $f(x)$ and $g(x)$.

$$\mathbb{R} \rightarrow \boxed{f(x) = x - 5} \rightarrow \mathbb{R} \qquad \{x \in \mathbb{R}: x \neq 1\} \rightarrow \boxed{g(x) = \frac{1}{x-1}} \rightarrow \{y \in \mathbb{R}: y \neq 0\}$$

For $g(x)$ to cope with the output from $f(x)$ we must ensure that the output does not include 1. Hence we must exclude 6 from the domain of $f(x)$.

Thus $g \circ f(x)$ has natural domain $\{x \in \mathbb{R}: x \neq 6\}$ and range $\{y \in \mathbb{R}: y \neq 0\}$.

b First identify the natural domain and range of $g(x)$ and $f(x)$.

$$\{x \in \mathbb{R}: x \neq 1\} \rightarrow \boxed{g(x) = \frac{1}{x-1}} \rightarrow \{y \in \mathbb{R}: y \neq 0\} \qquad \mathbb{R} \rightarrow \boxed{f(x) = x - 5} \rightarrow \mathbb{R}$$

$f(x)$ can cope with the output from $g(x)$ but note that 0 will not be in this output. Thus -5 will not be in the output from $f(x)$.

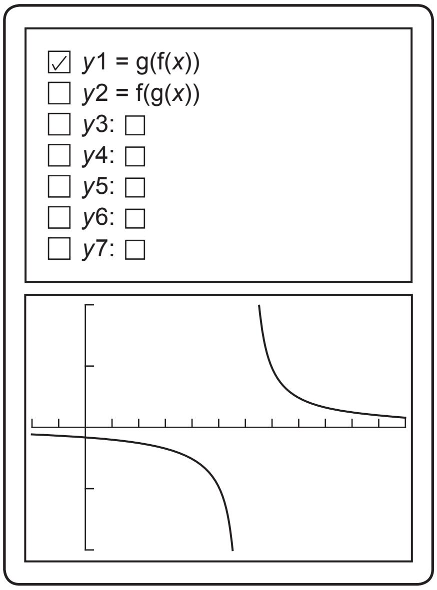
Thus $f \circ g(x)$ has natural domain $\{x \in \mathbb{R}: x \neq 1\}$ and range $\{y \in \mathbb{R}: y \neq -5\}$.

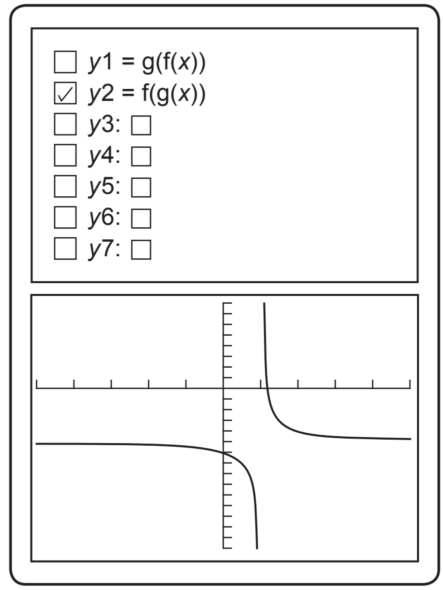
ISBN 9780170395274

The displays below confirm these domains and ranges for the composite functions $g \circ f(x)$ and $f \circ g(x)$, with $f(x)$ and $g(x)$ as defined on the right.

I.e., $g \circ f(x)$ has natural domain $\{x \in \mathbb{R}: x \neq 6\}$
and range $\{y \in \mathbb{R}: y \neq 0\}$.
$f \circ g(x)$ has natural domain $\{x \in \mathbb{R}: x \neq 1\}$
and range $\{y \in \mathbb{R}: y \neq -5\}$

Define $f(x) = x - 5$
done
Define $g(x) = \dfrac{1}{x - 1}$
done





We could have approached the previous example by expressing each composite function in terms of x and determining the domain and range of the resulting expression:

a
$$g \circ f(x) = g(x-5) = \frac{1}{(x-5)-1} = \frac{1}{(x-6)}$$

Domain $\{x \in \mathbb{R}: x \neq 6\}$
Range $\{y \in \mathbb{R}: y \neq 0\}$

b
$$f \circ g(x) = f\left(\frac{1}{x-1}\right) = \frac{1}{x-1} - 5$$

Domain $\{x \in \mathbb{R}: x \neq 1\}$
Range $\{y \in \mathbb{R}: y \neq -5\}$

However this approach must be used with caution. In some cases the fact that the final expression has come from a combination of functions means that the domain and range will not be the same as the final expression considered in isolation. The next example, in which

$$f(x) = \sqrt{x},\ g(x) = x^2 \quad \text{and} \quad g \circ f(x) = g(\sqrt{x}) = x,$$

is of this type.

ISBN 9780170395274

EXAMPLE 4

Determine the natural domain and the corresponding range of $g \circ f(x)$ given that

$$f(x) = \sqrt{x} \text{ and } g(x) = x^2.$$

Solution

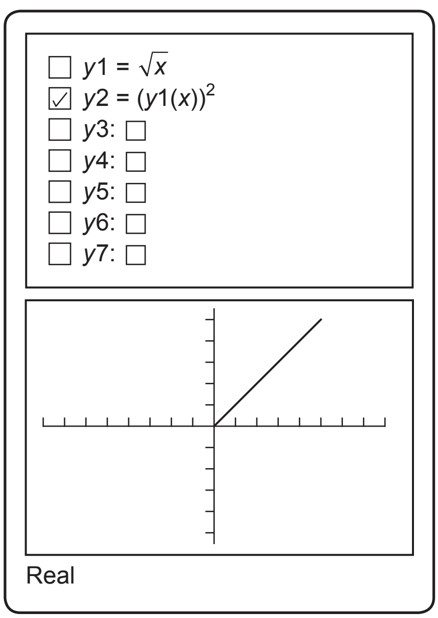
First identify the natural domain and range of $f(x)$ and $g(x)$.

$\{x \in \mathbb{R}: x \geq 0\} \rightarrow$ $\boxed{f(x) = \sqrt{x}}$ $\rightarrow \{y \in \mathbb{R}: y \geq 0\}$ $\quad\quad$ $\mathbb{R} \rightarrow$ $\boxed{g(x) = x^2}$ $\rightarrow \{y \in \mathbb{R}: y \geq 0\}$

$g(x)$ can cope with the output from $f(x)$ but note that the output from $g(x)$ consists of only the non-negative numbers.

Thus $g \circ f(x)$ has natural domain $\{x \in \mathbb{R}: x \geq 0\}$ and range $\{y \in \mathbb{R}: y \geq 0\}$.

The display below confirms this domain and range of $g \circ f(x)$ with $f(x) = \sqrt{x}$
and $g(x) = x^2$,
(and real values only).



Note that for real values, this domain and range is not what we would have obtained had we wrongly considered $g \circ f(x)$ to be the same as $h(x) = x$ which has domain $\mathbb{R}$ and range $\mathbb{R}$.

ISBN 9780170395274

EXAMPLE 5

Given that $f(x) = 2x$ and $g(x) = \sqrt{x-2}$ explain why $f[g(x)]$ is a function for the natural domain of $g(x)$ whereas $g[f(x)]$ is not a function for the natural domain of $f(x)$.

Solution

$\{x \in \mathbb{R}: x \geq 2\} \rightarrow$ [$g(x) = \sqrt{x-2}$] $\rightarrow \{y \in \mathbb{R}: y \geq 0\}$ $\qquad$ $\mathbb{R} \rightarrow$ [$f(x) = 2x$] $\rightarrow \mathbb{R}$

$f(x)$ can cope with the all the numbers in the range of $g(x)$ because the range of $g(x)$ is contained within the domain of $f(x)$.
Thus $f[g(x)]$ is a function for the natural domain of $g(x)$.

$\mathbb{R} \rightarrow$ [$f(x) = 2x$] $\rightarrow \mathbb{R}$ $\qquad$ $\{x \in \mathbb{R}: x \geq 2\} \rightarrow$ [$g(x) = \sqrt{x-2}$] $\rightarrow \{y \in \mathbb{R}: y \geq 0\}$

There are some numbers in the range of $f(x)$ that $g(x)$ will not be able to cope with. In this case the range of $f(x)$ contains elements that are outside the domain of $g(x)$.
Thus $g[f(x)]$ is not a function for the natural domain of $f(x)$.

Note: $g[f(x)]$ is a function if we restrict the domain of $f(x)$ to give an output that $g(x)$ can cope with.
Thus the natural domain of $g[f(x)]$ is $\{x \in \mathbb{R}: x \geq 1\}$ with corresponding range $\{y \in \mathbb{R}: y \geq 0\}$.

Exercise 3A

1 With $f(x) = x + 1$, $g(x) = 2x - 3$ and with an initial domain of $\{0, 1, 2, 3, 4\}$, determine the range of

- **a** $gf(x)$,
- **b** $fg(x)$,
- **c** $gg(x)$.

2 With $f(x) = x + 3$, $g(x) = (x - 1)^2$, $h(x) = x^3$, and with $\{1, 2, 3\}$ as the initial domain, determine the range of

- **a** $gf(x)$,
- **b** $fgh(x)$,
- **c** $hgf(x)$.

3 If $f(x) = x + 5$ and $g(x) = x - 5$ determine the natural domain and range of each of the following.

- **a** $f(x)$
- **b** $g(x)$
- **c** $f(x) + g(x)$
- **d** $f(x) - g(x)$
- **e** $f(x) \cdot g(x)$
- **f** $\dfrac{f(x)}{g(x)}$

ISBN 9780170395274

4 Given that $f(x) = 3x + 2$, $g(x) = \frac{2}{x}$ and $h(x) = \sqrt{x}$, express each of the following functions in terms of some or all of f, g and h.

a $\frac{2}{3x+2}$ **b** $\sqrt{3x+2}$ **c** $\frac{6}{x} + 2$

d $3\sqrt{x} + 2$ **e** $\frac{2}{\sqrt{x}}$ **f** $\sqrt{\frac{2}{x}}$

g $9x + 8$ **h** $x^{0.25}$ **i** $27x + 26$

5 Given that $f(x) = 2x - 3$, $g(x) = 4x + 1$ and $h(x) = x^2 + 1$, express each of the following functions in a similar way (i.e. in terms of x), simplifying where possible.

a $f \circ f(x)$ **b** $g \circ g(x)$ **c** $h \circ h(x)$

d $f \circ g(x)$ **e** $g \circ f(x)$ **f** $f \circ h(x)$

g $h \circ f(x)$ **h** $g \circ h(x)$ **i** $h \circ g(x)$

6 Given that $f(x) = 2x + 5$, $g(x) = 3x + 1$ and $h(x) = 1 + \frac{2}{x}$, express each of the following functions in a similar way (i.e. in terms of x), simplifying where possible.

a $f \circ f(x)$ **b** $g \circ g(x)$ **c** $h \circ h(x)$

d $f \circ g(x)$ **e** $g \circ f(x)$ **f** $f \circ h(x)$

g $h \circ f(x)$ **h** $g \circ h(x)$ **i** $h \circ g(x)$

For each of questions 7 to 12, $g[f(x)]$ is not a function for the natural domain of $f(x)$. State the minimal restriction necessary on the natural domain of $f(x)$ for $g[f(x)]$ to be defined for this domain.

7 $f(x) = x - 4$, $g(x) = \sqrt{x}$

8 $f(x) = 4 - x$, $g(x) = \sqrt{x}$

9 $f(x) = 4 - x^2$, $g(x) = \sqrt{x}$

10 $f(x) = 4 - |x|$, $g(x) = \sqrt{x}$

11 $f(x) = x + 3$, $g(x) = \sqrt{x - 5}$

12 $f(x) = x - 6$, $g(x) = \sqrt{x + 3}$

13 If $f(x) = x^2 + 3$ and $g(x) = \frac{1}{x}$ find:

a $f(3)$ **b** $f(-3)$ **c** $g(2)$ **d** $fg(1)$ **e** $gf(1)$

f the natural domain and corresponding range of f

g the natural domain and corresponding range of g

h the natural domain and corresponding range of $gf(x)$

(Check your answer using a graphic calculator with Y1 = X^2 + 3 and Y2 = 1 ÷ Y1.)

i the natural domain and corresponding range of $fg(x)$

ISBN 9780170395274

14 If $f(x) = 25 - x^2$ and $g(x) = \sqrt{x}$ find:

a $f(5)$ **b** $f(-5)$ **c** $g(4)$ **d** $fg(4)$ **e** $gf(4)$

f the natural domain and corresponding range of f

g the natural domain and corresponding range of g

h the natural domain and corresponding range of $gf(x)$

i the natural domain and corresponding range of $fg(x)$

15 State the natural domain and the corresponding range of each of the following functions given that $f(x) = x + 2$ and $g(x) = \dfrac{1}{x-3}$.

a $g \circ f(x)$ **b** $f \circ g(x)$

16 State the natural domain and the corresponding range of each of the following functions given that $f(x) = \sqrt{x}$ and $g(x) = 2x - 1$.

a $g \circ f(x)$ **b** $f \circ g(x)$

17 State the natural domain and the corresponding range of each of the following functions given that $f(x) = \dfrac{1}{x^2}$ and $g(x) = \sqrt{x}$.

a $g \circ f(x)$ **b** $f \circ g(x)$

18 Given that $f(x) = x + 3$ and $g(x) = \sqrt{x}$ explain why $f[g(x)]$ is a function for the natural domain of $g(x)$ whereas $g[f(x)]$ is not a function for the natural domain of $f(x)$.

19 Given that $f(x) = x + 3$ and $g(x) = \dfrac{1}{x-5}$ explain why $f[g(x)]$ is a function for the natural domain of $g(x)$ whereas $g[f(x)]$ is not a function for the natural domain of $f(x)$.

20 State the natural domain and the corresponding range of each of the following functions given that $f(x) = x^2 - 9$ and $g(x) = \dfrac{1}{x}$.

a $g \circ f(x)$ **b** $f \circ g(x)$

ISBN 9780170395274

Inverse functions

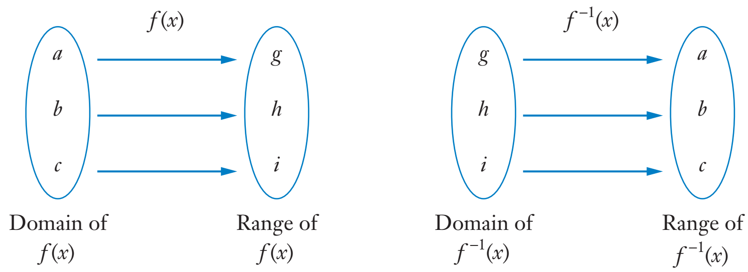
WS Inverse functions

If a function $f(x)$ maps the domain $\{a, b, c\}$ onto the range $\{g, h, i\}$ such that

$$f(a) = g \qquad f(b) = h \qquad f(c) = i$$

then the inverse function, $f^{-1}(x)$, will map $\{d, e, f\}$ back to $\{a, b, c\}$ such that

$$f^{-1}(g) = a \qquad f^{-1}(h) = b \qquad f^{-1}(i) = c$$



Thus	**the range of $f(x)$ is the domain of $f^{-1}(x)$**
and	**the domain of $f(x)$ is the range of $f^{-1}(x)$.**

To determine the inverse of a function we can

- construct the function as a sequence of steps and then reverse the process: see method one in the next example,

or

- rearrange the function rule: see method two in the next example.

EXAMPLE 6

Find the inverse of the function $f(x) = 2x + 3$.

Solution

Method One: Reversing the flow chart

Write the function as a flow chart with input x and output $2x + 3$.

$x \rightarrow \boxed{\times 2} \rightarrow \boxed{+3} \rightarrow 2x + 3$

Reverse the flow chart, writing the inverse of each operation.

$\leftarrow \boxed{\div 2} \leftarrow \boxed{-3} \leftarrow$

For an input of x this reversed flow chart will output the inverse function.

$\dfrac{x-3}{2} \leftarrow \boxed{\div 2} \leftarrow \boxed{-3} \leftarrow x$

Thus $f^{-1}(x) = \dfrac{x-3}{2}$. [Check: $f(5) = 13, f^{-1}(13) = 5$.]

Method Two: Rearranging the formula

If
$$\begin{aligned} y &= 2x+3 \\ y-3 &= 2x \end{aligned}$$
and so
$$x = \frac{y-3}{2}$$

Thus given y we can output x using $x = \dfrac{y-3}{2}$

$$\therefore \quad f^{-1}(x) = \frac{x-3}{2}$$

solve($y = 2x + 3$, x)

$$\left\{x = \frac{y}{2} - \frac{3}{2}\right\}$$

EXAMPLE 7

Find the inverse of the function $f(x) = 1 + \dfrac{1}{3+x}$

Solution

Method One: Reversing the flow chart

Write the function as a flow chart with input x and output $1 + \dfrac{1}{3+x}$.

x → [+ 3] → [Invert] → [+ 1] → $\dfrac{1}{x+3} + 1$

Reverse the flow chart, writing the inverse of each operation.

← [− 3] ← [Invert] ← [− 1] ←

For an input of x this reversed flow chart will output the inverse function.

$\dfrac{1}{x-1} - 3$ ← [− 3] ← [Invert] ← [− 1] ← x

Thus $f^{-1}(x) = \dfrac{1}{x-1} - 3$.   [Check: $f(2) = 1.2, f^{-1}(1.2) = 2$.]

Method Two: Rearranging the formula

If
$$y = 1 + \frac{1}{3+x}$$
then
$$\begin{aligned} y-1 &= \frac{1}{3+x} \\ 3+x &= \frac{1}{y-1} \\ x &= \frac{1}{y-1} - 3 \end{aligned}$$

solve$\left(y = 1 + \dfrac{1}{3+x}, x\right)$

$$\left\{x = \frac{-3 \cdot y}{y-1} + \frac{4}{y-1}\right\}$$

Thus given y we can output x using $x = \dfrac{1}{y-1} - 3$

$$\therefore \quad f^{-1}(x) = \frac{1}{x-1} - 3$$

ISBN 9780170395274

Graphical relationship between a function and its inverse

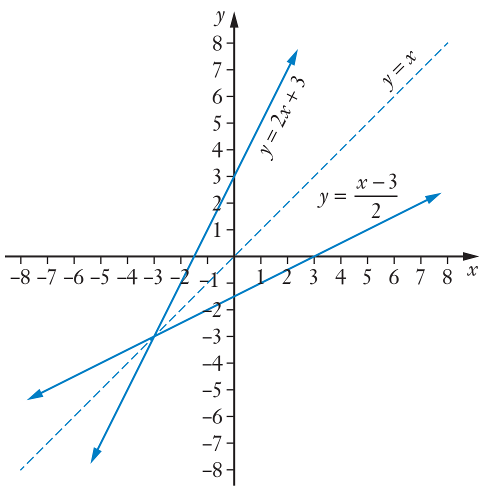
Consider some function $f(x)$ and its inverse function $f^{-1}(x)$.

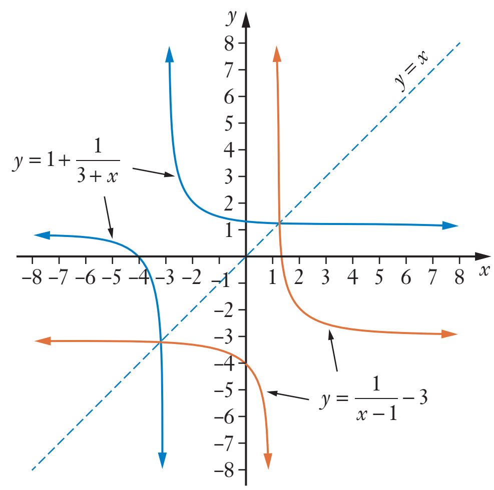
For every point (a, b) that exists on the graph of $y = f(x)$ there will exist a point (b, a) on the graph of $y = f^{-1}(x)$.

Thus the graph of $y = f^{-1}(x)$ will be that of $y = f(x)$ reflected in the line $y = x$.

This is illustrated below for the functions of the previous two examples:

$$f(x) = 2x + 3, f^{-1}(x) = \frac{x-3}{2} \qquad \text{and} \qquad g(x) = 1 + \frac{1}{3+x},\ g^{-1}(x) = \frac{1}{x-1} - 3.$$





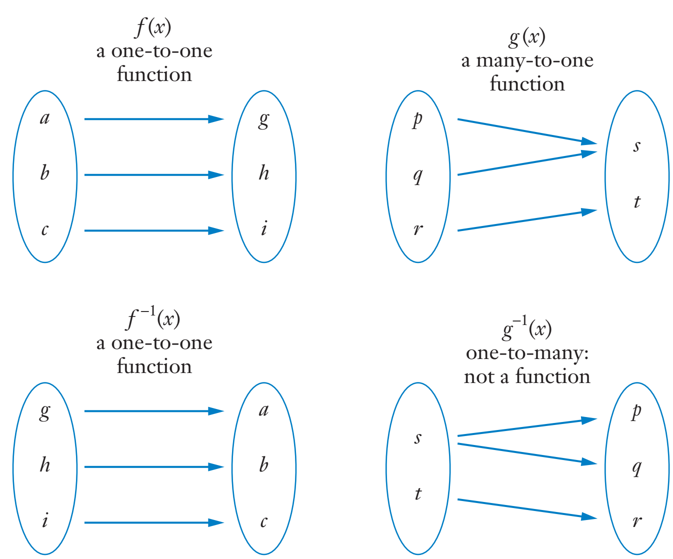
Condition for the inverse to exist as a function

One-to-one functions

Under our requirement that a function maps each element of the domain onto one and only one element of the range it follows that only one-to-one functions can have inverses that are functions.

Thus for $f(x)$ to have an inverse that is a function, the graph of $y = f(x)$ must pass the horizontal line test.

If the graph of the function is such that any horizontal line placed on the graph cuts the function no more than once, the function will be one-to-one and will therefore have an inverse function.



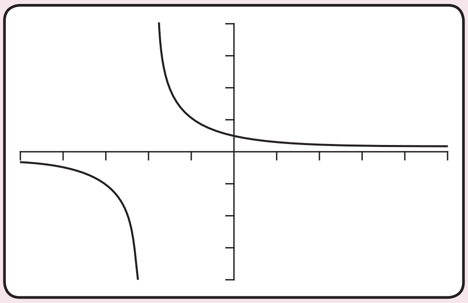
ISBN 9780170395274

EXAMPLE 8

If $f(x) = \dfrac{1}{x+2}$ state whether the inverse function, $f^{-1}(x)$, exists. If it does exist, determine a formula for $f^{-1}(x)$ and state its domain and range.

Solution

First view the graph of the function.



$f(x)$ passes the horizontal line test and is therefore one-to-one for its natural domain $\{x \in \mathbb{R}: x \neq -2\}$, and has range $\{y \in \mathbb{R}: y \neq 0\}$.

Thus $f^{-1}(x)$ exists and has domain given by $\{x \in \mathbb{R}: x \neq 0\}$ and range $\{y \in \mathbb{R}: y \neq -2\}$.

If $\quad y = \dfrac{1}{x+2} \quad$ then $\quad x = \dfrac{1}{y} - 2$

$\therefore \quad f^{-1}(x) = \dfrac{1}{x} - 2.$

Domain $\{x \in \mathbb{R}: x \neq 0\}$
and range $\{y \in \mathbb{R}: y \neq -2\}$.

Note

Some calculators can display the inverse relationship automatically. See if your calculator can, and if so use it to confirm the domain and range for the above example.

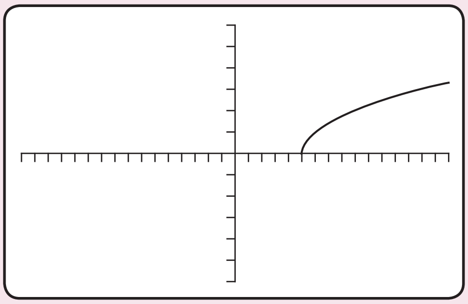
However care needs to be taken in interpreting the display as being the inverse *function*. Some calculators display the inverse relationship and this may not itself be a function.

EXAMPLE 9

If $f(x) = \sqrt{x-5}$ determine whether the inverse function, $f^{-1}(x)$, exists. If it does exist, determine a formula for $f^{-1}(x)$ and state its domain and range.

Solution

First view the graph of the function.



$f(x)$ passes the horizontal line test and is therefore one-to-one for its natural domain, $\{x \in \mathbb{R}: x \geq 5\}$ and has range $\{y \in \mathbb{R}: y \geq 0\}$.

Thus $f^{-1}(x)$ exists and has domain given by $\{x \in \mathbb{R}: x \geq 0\}$ and range $\{y \in \mathbb{R}: y \geq 5\}$.

If $\quad y = \sqrt{x-5} \quad$ then $\quad x = y^2 + 5$

$\therefore \quad f^{-1}(x) = x^2 + 5.$ $\quad$ Domain $\{x \in \mathbb{R}: x \geq 0\}$ and range $\{y \in \mathbb{R}: y \geq 5\}$.

If a function is not one-to-one we can restrict the domain of the function to one in which the function is one-to-one and then an inverse function can exist.

ISBN 9780170395274

EXAMPLE 10

The function $f(x)$ is defined by $f(x) = (x - 1)^2$.

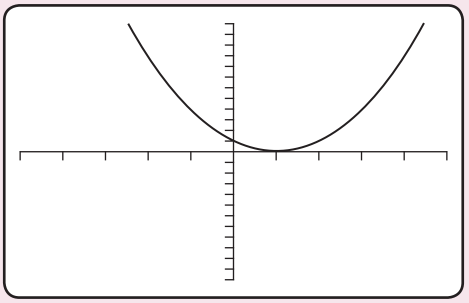
a Explain why $f(x)$ cannot have an inverse function for its natural domain.

b State a suitable restriction for the domain of $f(x)$ so that, with this restriction applied, $f(x)$ can have an inverse function.

c Determine this inverse function and state its domain and range.

Solution

a First view the graph of the function.



$f(x)$ has natural domain $\mathbb{R}$ and range $\{y \in \mathbb{R}: y \geq 0\}$.

The function is not one-to-one for this domain.

e.g. $f(0) = 1$ and $f(2) = 1$.

Thus $f(x)$ cannot have an inverse function for its natural domain.

b If we restrict the domain of $f(x)$ to $x \geq 1$, or $x \leq 1$, the function is one-to-one. Thus either of these would be suitable restrictions on the domain.

c If $y = (x-1)^2$ then $\pm\sqrt{y} = x - 1$

$\therefore x = 1 \pm \sqrt{y}$

Thus the inverse function $f^{-1}(x)$ will be either $1+\sqrt{x}$ or $1-\sqrt{x}$.

For $f(x) = (x-1)^2$, with domain $\{x \in \mathbb{R}: x \geq 1\}$ range $\{y \in \mathbb{R}: y \geq 0\}$

$f^{-1}(x) = ???$ with domain $\{x \in \mathbb{R}: x \geq 0\}$ range $\{y \in \mathbb{R}: y \geq 1\}$

For the given range it follows that ??? must be $1+\sqrt{x}$ not $1-\sqrt{x}$.

(This also follows if we consider the reflection of $f(x)$, $x \geq 1$, in $y = x$.)

Thus $f^{-1}(x) = 1+\sqrt{x}$ with domain $\{x \in \mathbb{R}: x \geq 0\}$ range $\{y \in \mathbb{R}: y \geq 1\}$

Alternatively:

For $f(x) = (x-1)^2$, with domain $\{x \in \mathbb{R}: x \leq 1\}$ range $\{y \in \mathbb{R}: y \geq 0\}$

$f^{-1}(x) = 1-\sqrt{x}$ with domain $\{x \in \mathbb{R}: x \geq 0\}$ range $\{y \in \mathbb{R}: y \leq 1\}$.

ISBN 9780170395274

Exercise 3B

1 Which of the following functions have inverse functions on their natural domains?

a $f(x) = x$ **b** $f(x) = 2x + 3$ **c** $f(x) = 5x - 3$

d $f(x) = x^2$ **e** $f(x) = (2x - 1)^2$ **f** $f(x) = x^2 + 4$

g $f(x) = \dfrac{1}{x}$ **h** $f(x) = \dfrac{1}{x - 3}$ **i** $f(x) = \dfrac{1}{x^2}$

Find an expression for the inverse function of each of the following and state its domain and range.

2 $f(x) = x - 2$ **3** $f(x) = 2x - 5$ **4** $f(x) = 5x + 2$

5 $f(x) = \dfrac{1}{x - 4}$ **6** $f(x) = \dfrac{1}{x + 3}$ **7** $f(x) = \dfrac{1}{2x - 5}$

8 $f(x) = 1 + \dfrac{1}{2 + x}$ **9** $f(x) = 3 - \dfrac{1}{x - 1}$ **10** $f(x) = 4 + \dfrac{2}{2x - 1}$

11 $f(x) = \sqrt{x}$ **12** $f(x) = \sqrt{x + 1}$ **13** $f(x) = \sqrt{2x - 3}$

WS Composition of functions

Given that $f(x) = 2x + 5$, $g(x) = 3x + 1$ and $h(x) = 1 + \dfrac{2}{x}$ express each of the following functions in this form (i.e. as expressions in terms of x).

14 $f^{-1}(x)$ **15** $g^{-1}(x)$ **16** $h^{-1}(x)$

17 $f \circ f^{-1}(x)$ **18** $f^{-1} \circ f(x)$ **19** $f \circ h^{-1}(x)$

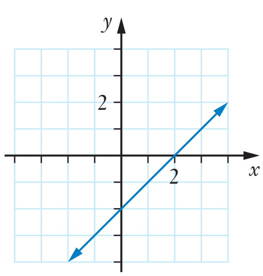
20 $(f \circ g)^{-1}(x)$ **21** $g^{-1} \circ f^{-1}(x)$ **22** $f \circ g^{-1}(x)$

23 For each of the following state whether the graph shown is a function.

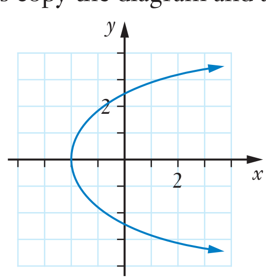
For those that are functions state whether they are one-to-one or not.

For those that are one-to-one functions copy the diagram and add the graph of the inverse function.

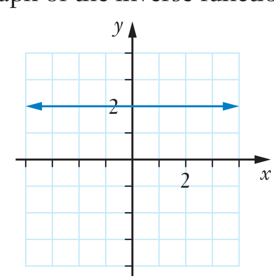
a



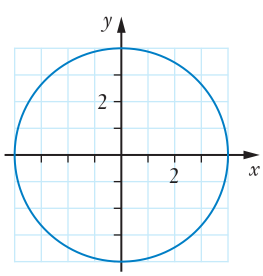
b



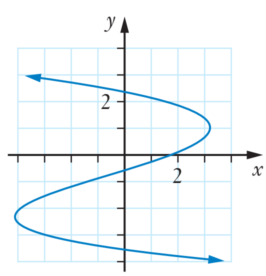
c



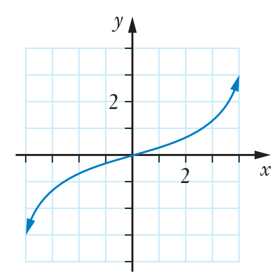
d



e



f

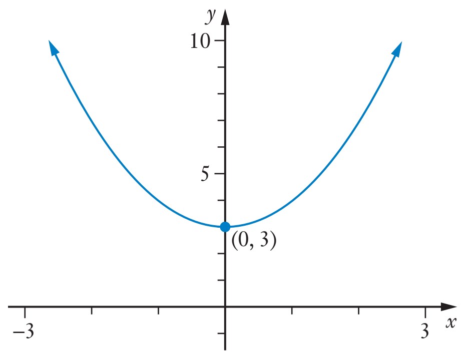


ISBN 9780170395274

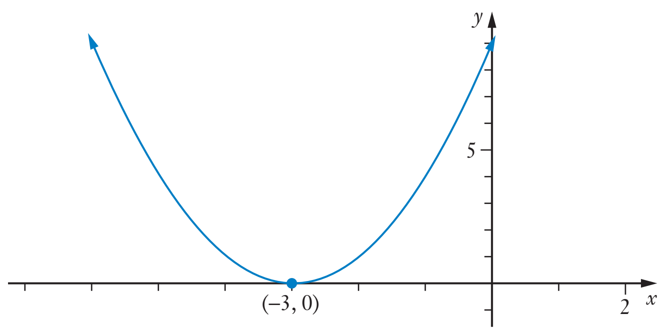
Each of questions **24** to **27** shows the graph of a function.

For each one, state a suitable restriction to the domain of $f(x)$ so $f^{-1}(x)$ exists as a function (do not restrict the domain any more than is necessary) and state the inverse function together with its domain and range.

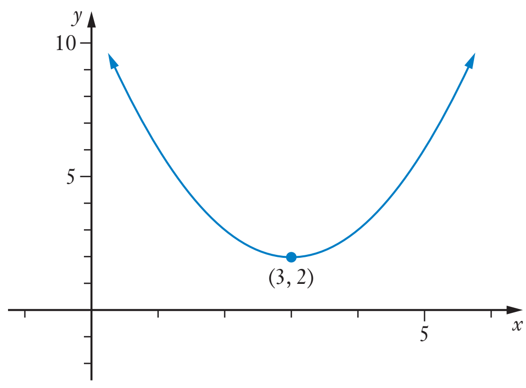
24 $f(x) = x^2 + 3$



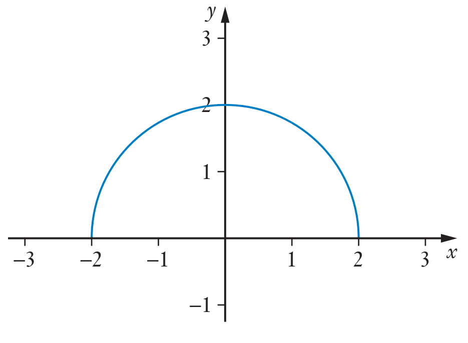
25 $f(x) = (x + 3)^2$



26 $f(x) = (x - 3)^2 + 2$



27 $f(x) = \sqrt{4 - x^2}$



ISBN 9780170395274

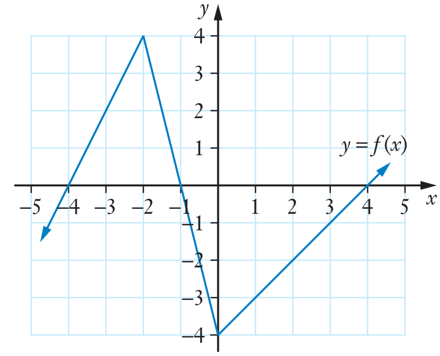
The absolute value function

Consider the function $f(x)$ shown on the right.

Suppose we want to graph $\qquad y = |f(x)|$

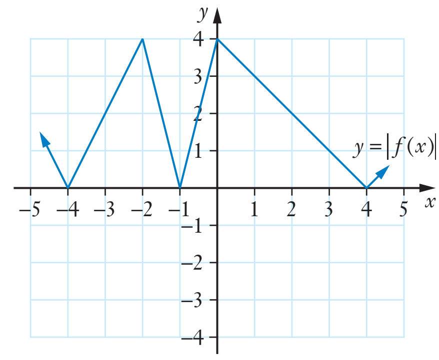
Wherever $f(x) \geq 0$ then $\qquad |f(x)| = f(x)$

So, any part of the graph of $f(x)$ that lies above the x-axis will also feature on the graph of $|f(x)|$.

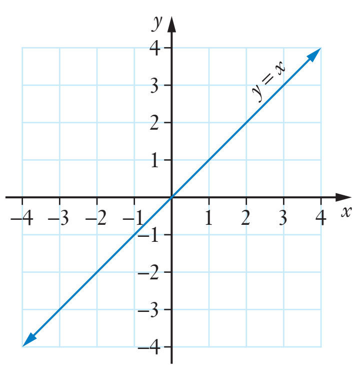


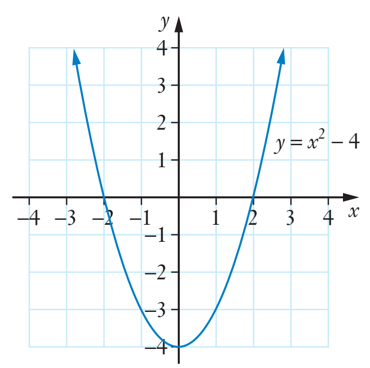
Any part lying below the x-axis indicates that $f(x)$ is negative for these x values. Taking the absolute value will make these values positive, i.e. taking the absolute value of $f(x)$ will reflect, in the x-axis, any parts of $f(x)$ lying below the x-axis.

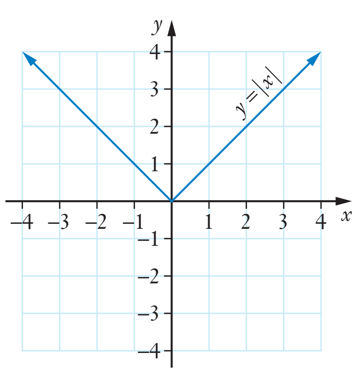
Hence the graph of $y = |f(x)|$ will be as shown on the right.

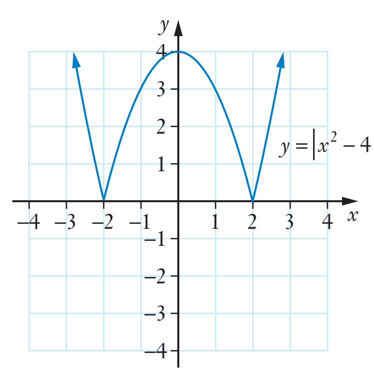


Similar considerations of the graphs of $y = x$ and $y = x^2 - 4$, for example, allow us to draw the graphs of $y = |x|$ and $y = |x^2 - 4|$, as shown below.









ISBN 9780170395274

The absolute value of x as $\sqrt{x^2}$

Earlier in this chapter, when considering using the output from one function as the input of another function, we considered the composite function $g[f(x)]$ for $f(x) = \sqrt{x}$ and $g(x) = x^2$.

We found that $g[f(x)]$ has natural domain $\{x \in \mathbb{R}: x \geq 0\}$ and range $\{y \in \mathbb{R}: y \geq 0\}$.

Now let us instead consider $f[g(x)]$, for these same two functions.

First identify the natural domain and range of $g(x)$ and $f(x)$.

$\mathbb{R} \rightarrow$ [$g(x) = x^2$] $\rightarrow \{y \in \mathbb{R}: y \geq 0\}$ $\qquad$ $\{x \in \mathbb{R}: x \geq 0\} \rightarrow$ [$f(x) = \sqrt{x}$] $\rightarrow \{y \in \mathbb{R}: y \geq 0\}$

$f(x)$ can cope with the output from $g(x)$ but note that the output from $f(x)$ consists of only the non negative numbers.

Thus $f[g(x)]$ has natural domain $\mathbb{R}$ and range $\{y \in \mathbb{R}: y \geq 0\}$.

Formulating an algebraic rule for $f[g(x)]$:

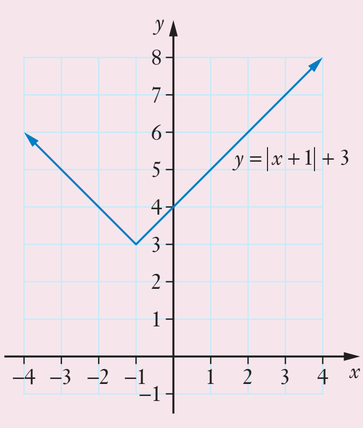
$$\begin{aligned} f[g(x)] &= f[x^2] \\ &= \sqrt{x^2} \\ &= \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x \leq 0. \end{cases} \\ &= |x| \end{aligned}$$

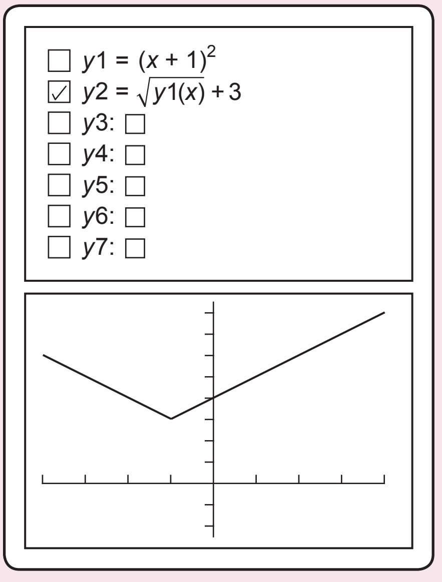
EXAMPLE 11

If $f(x) = \sqrt{x} + 3$ and $g(x) = (x + 1)^2$, draw the graph of $f[g(x)]$.

Solution

$$\begin{aligned} f[g(x)] &= f[(x+1)^2] \\ &= \sqrt{(x+1)^2} + 3 \\ &= |x+1| + 3 \end{aligned}$$

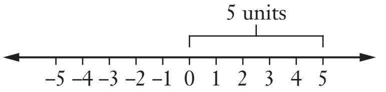




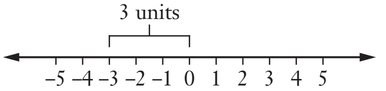
Solving equations involving absolute values

The *Preliminary work* section at the beginning of this book reminded us that the absolute value of a number x, written $|x|$, is the distance the number is from the origin. For example:

$|5|$ is the distance from the point 5 to the origin, i.e. 5.



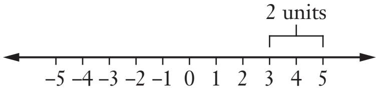
$|-3|$ is the distance from the point –3 to the origin, i.e. 3.



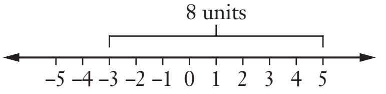
Similarly $|x - a|$ tells us the distance x is from the point a.

For example:

$|(5)-(3)|$ is the distance from the point 5 to the point 3, i.e. 2.



$|(5)+(3)|$, i.e. $|(5)-(-3)|$, is the distance from the point 5 to the point –3. i.e. 8.



Note

- If $|x| = a\ (a \geq 0)$, then x is a units from the origin.
  Thus either $x = a$
  or $x = -a$
- If $|x| = |y|$ then x and y are equidistant from the origin.
  Thus either $x = y$
  or $x = -y$

Asked to solve equations like $|x+2| = 5$ or $|x-2| = |x+6|$, without the assistance of a calculator, we could proceed in a number of ways:

- Algebraically
- Using the number line
- Graphically

Note

Whilst the syllabus for this unit does not specifically mention solving equations involving absolute value functions, the topic is included here to further an understanding of the graphs of absolute value functions and to meet the syllabus requirement of being able to use and apply the absolute value of a real number.

ISBN 9780170395274

EXAMPLE 12

Determine the values of x for which $|x + 2| = 5$.

Solution

Algebraically

Either $x + 2 = 5$ or $x + 2 = -5$

$\therefore$ $x = 3$ or $x = -7$

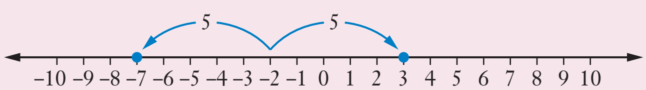
The required values are $x = -7$ and $x = 3$.

Using the number line

$|x + 2| = 5$

i.e $|x - (-2)| = 5$.

Thus x is 5 units from -2.



The required values are $x = -7$ and $x = 3$.

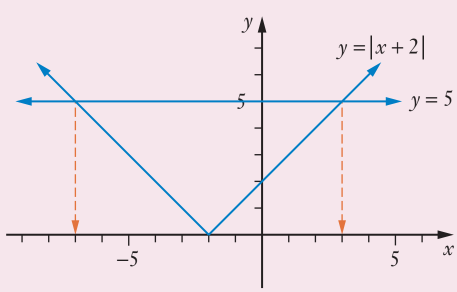
Graphically

Given the graphs of $y = |x + 2|$

and $y = 5$

the intersection of these lines is where $|x + 2| = 5$

The required values are $x = -7$ and $x = 3$.



The values determined above can be confirmed using the ability of some calculators to solve equations involving absolute values.

solve($|x + 2| = 5$, x)

$\{x = -7, x = 3\}$

ISBN 9780170395274

EXAMPLE 13

Solve $|x-2| = |x+6|$.

Solution

Algebraically

Either	$(x-2)$	$=$	$(x+6)$	or	$(x-2)$	$=$	$-(x+6)$
$\therefore$	-2	$=$	6	or	$x-2$	$=$	$-x-6$
	No solution			or	x	$=$	-2

The only solution is $x = -2$.

Alternatively, squaring both sides of the equation:

$$(x-2)^2 = (x+6)^2$$
$$x^2 - 4x + 4 = x^2 + 12x + 36$$
$$-16x = 32$$

i.e. $x = -2$

Check: If $x = -2$ then $|x-2| = 4$
and $|x+6| = 4$, as required. (See note below.)

Note: The final check is necessary because squaring both sides of an equation can introduce 'false' solutions.

For example consider the equation $x = 2$.

Squaring gives $x^2 = 4$
which has solutions $x = \pm 2$.

However -2 is clearly not a solution to the equation $x = 2$.

Thus if the technique of 'squaring both sides' is used the validity of the solutions must be checked.

Using the number line

$$|x-2| = |x+6|$$

i.e. $|x-2| = |x-(-6)|$.

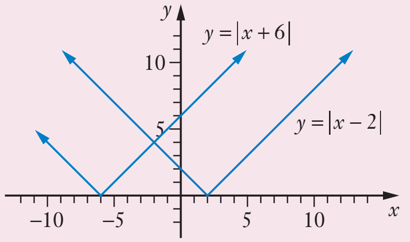
The distance from x to the point 2 is the same as from x to the point -6.
Thus x must be midway between -6 and 2.
The only solution is $x = -2$.

Graphically

Considering the graphs of $y = |x-2|$
and $y = |x+6|$
the intersection of these lines is where $|x-2| = |x+6|$

The only solution is $x = -2$.

- Again the solution obtained above can be confirmed using the solve facility of some calculators.



 ISBN 9780170395274

Some equations involving absolute values, e.g. $|x-2| = 2x-7$, can be difficult to interpret from the idea of distances on a number line. These equations can also need special care if an algebraic approach is used. Indeed if we have to demonstrate our ability to solve such equations without simply using the solve facility on a calculator, the graphical approach is probably the one least likely to cause errors. It may not be the quickest method but you will probably reduce the number of errors you are likely to make. In the next example, algebraic and graphical approaches are shown.

EXAMPLE 14

Determine the values of x for which $|x-2| = 2x-7$.

Solution

Algebraically

- We could proceed as in example 13 and then check that the apparent solutions are valid:

				or			
	$x-2$	$=$	$2x-7$	or	$-(x-2)$	$=$	$2x-7$
$\therefore$	5	$=$	x	or	x	$=$	3
Check:	$\lvert 5-2\rvert$	$=$	$2(5)-7$		$\lvert 3-2\rvert$	$\neq$	$2(3)-7$

The only solution is $x = 5$.

- We could square both sides:

$$\begin{aligned}(x-2)^2 &= (2x-7)^2\\ x^2-4x+4 &= 4x^2-28x+49\\ 0 &= 3x^2-24x+45\\ \text{i.e.}\quad x^2-8x+15 &= 0\end{aligned}$$

Giving	x	$=$	3	or	x	$=$	5
Check:	$\lvert 3-2\rvert$	$\neq$	$2(3)-7$		$\lvert 5-2\rvert$	$=$	$2(5)-7$

The only solution is $x = 5$.

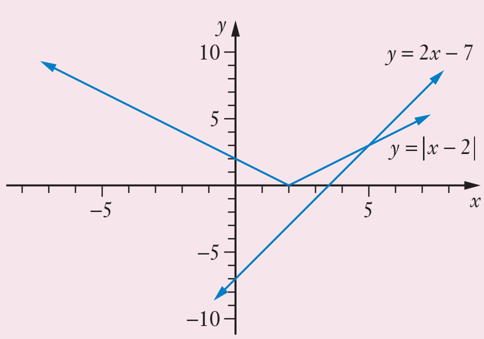
- Alternatively we could consider intervals of the number line separately:

If $x \geq 2$:	$x-2$	$=$	$2x-7$	If $x < 2$:	$-(x-2)$	$=$	$2x-7$
	x	$=$	5		x	$=$	3
This is consistent with	x	$\geq$	2	This is inconsistent with	x	$<$	2
$\therefore$	x	$=$	5	$\therefore$	x	$\neq$	3

The only solution is $x = 5$.

Graphically

Consider the graphs of $y = |x-2|$
and $y = 2x-7$
and determine their point of intersection.
The only solution is $x = 5$.



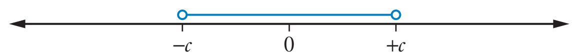
Absolute values

Solving inequalities involving absolute values

Check that you understand each of the following ideas.

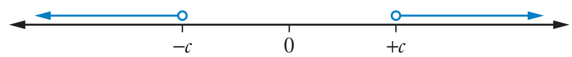
- If $|x| < c$, $c > 0$, then the distance from x to the origin is less than c,

  $\therefore \quad -c < x < c$



- If $|x| > c$, $c > 0$, then the distance from x to the origin is greater than c,

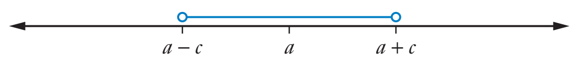
  $\therefore$ either $x < -c$ or $x > c$



- If $|x - a| < c$, $c > 0$, then the distance from x to the point a is less than c,

  $\therefore \quad -c < (x - a) < c$

  i.e. $\quad a - c < x < a + c$



As we did when solving *equations* involving absolute values, we will consider three ways of solving *inequalities* involving absolute values (other than using the solve facility of some calculators):

- Algebraic
- Using the number line
- Graphical

The following examples demonstrate these methods of solution.

You are encouraged to also explore the capability of your calculator to solve inequalities involving absolute values.

EXAMPLE 15

Determine the values of x for which $|x| > 4$.

Solution

Algebraically

If $\quad |x| > 4 \quad$ then either $\quad x > 4 \quad$ or $\quad x < -4$.

Thus the values of x are as shown:

This would be written: $\quad x < -4, x > 4$
or, using set notation: $\quad \{x \in \mathbb{R}: x < -4\} \cup \{x \in \mathbb{R}: x > 4\}$
where the symbol $\cup$ means the two sets are *united* to give the complete set of values.

Using the number line

If $\quad |x| > 4 \quad$ then x is more than 4 units from the origin.

Thus once again the values of x are:

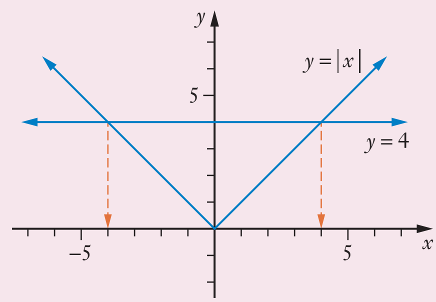
ISBN 9780170395274

Graphically

View the graphs of

$$y = |x| \qquad \text{and} \qquad y = 4$$

and determine the values of x for which the graph of $y = |x|$ lies 'above' that of $y = 4$.



Again the values of x are:

EXAMPLE 16

Determine the values of x for which $|x - 2| \le 5$.

Solution

Algebraically

If $|x - 2| \le 5$ then $-5 \le x - 2 \le 5$

$\therefore \quad -5 + 2 \le x \le 5 + 2$

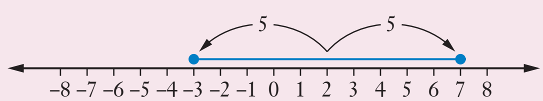
i.e. $-3 \le x \le 7$

Using the number line

If $|x - 2| \le 5$ then the distance from x to 2 is less than or equal to 5 units.

Thus the values of x are as shown:

i.e. $-3 \le x \le 7$, as before.



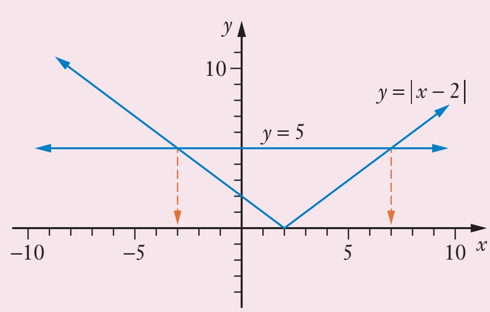
Graphically

Consider the graphs of $y = |x - 2|$

and $y = 5$

and determine the values of x for which the graph of $y = |x - 2|$ lies 'below' that of $y = 5$.

Thus again $-3 \le x \le 7$.



ISBN 9780170395274

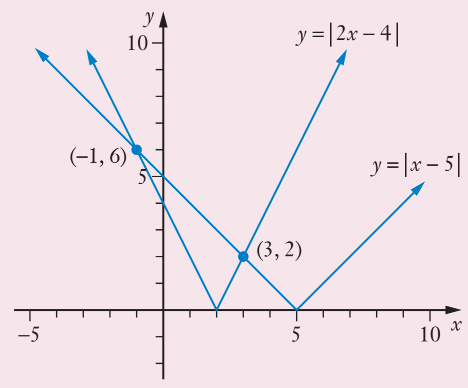
EXAMPLE 17

Show the graphs of $y = |2x - 4|$
and $y = |x - 5|$
on the same set of axes.

Hence determine the values of x for which $|2x - 4| \geq |x - 5|$.

Solution

The graphs of $y = |2x - 4|$
and $y = |x - 5|$
are shown on the right.



Looking for where the graph of $y = |2x - 4|$
is 'above' the graph of $y = |x - 5|$
we see that the inequality is true for:

$x \leq -1$ and for $x \geq 3$.

Exercise 3C

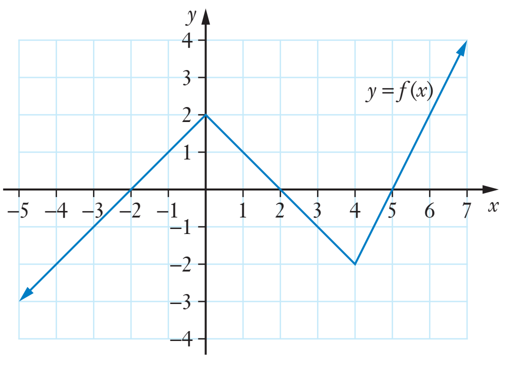
1 (Use an x-axis from -5 to 5.)
On graph paper, or squared paper, draw the graph of $y = |x + 1|$.

2 (Use an x-axis from -3 to 6.)
On graph paper, or squared paper, draw the graph of $y = |2x - 2|$.

3 (Use an x-axis from -3 to 7.)
On graph paper, or squared paper, draw the graphs of $y = |x - 2|$ and $y = 3 + |x - 2|$.

4 (Use an x-axis from -2 to 4.)
On graph paper, or squared paper, draw the graph of $y = |(x - 2)^2 - 1|$.

5 The graph of $y = f(x)$ is shown on the right.
Draw the graph of $y = |f(x)|$.



ISBN 9780170395274

6 (Use an x-axis from –3 to 5.)

For this question draw all three graphs on the same set of axes.

a Draw the graph of $y = |x|$.

b Draw the graph of $y = |x - 3|$.

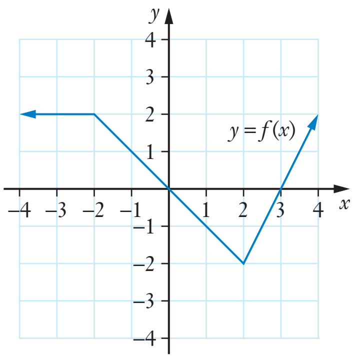
c Hence draw the graph of $y = |x| + |x - 3|$.

7 The graph of $y = f(x)$ is shown on the right.

Draw the graph of each of the following.

a $y = |f(x)|$

b $y = f(|x|)$

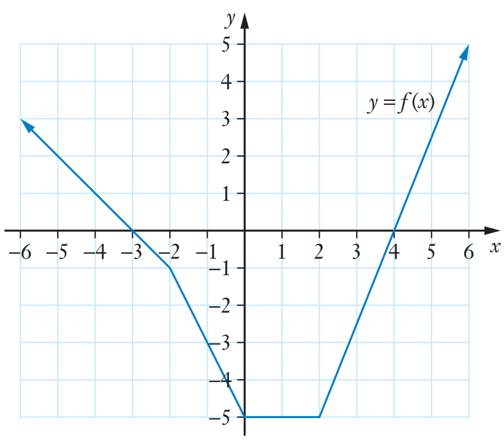


8 The graph of $y = f(x)$ is shown on the right.

Draw the graph of each of the following.

a $y = |f(x)|$

b $y = f(|x|)$



9 How will the graph of $y = g(|x|)$ relate to the graph of $y = g(x)$?

10 For this question $f(x) = 2 + \sqrt{x}$ and $g(x) = (x + 1)^2$.

a Explain why $f[g(x)]$ is defined for all real x, and state the corresponding range.

b Obtain an expression for $f[g(x)]$ in terms of x.

c Draw the graph of the composite function $f[g(x)]$.

11 For this question $f(x) = 1 - \sqrt{x}$ and $g(x) = (x - 2)^2$.

a Explain why $f[g(x)]$ is defined for all real x, and state the corresponding range.

b Obtain an expression for $f[g(x)]$ in terms of x.

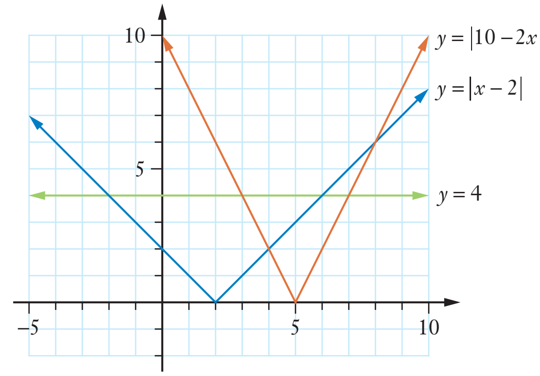
c Draw the graph of the composite function $f[g(x)]$.

12 Use the diagram on the right to solve each of the following equations:

a $|10-2x|=4$

b $|x-2|=4$

c $|10-2x|=|x-2|$



13 (Use an x-axis from −8 to 8 and a y-axis from −1 to 9.)

Draw the graphs of $y=5$, $y=|x|$, $y=3-0.5x$ and $y=|2x+3|$.

Hence solve the following equations.

a $|2x+3|=5$

b $3-0.5x=|x|$

c $3-0.5x=|2x+3|$

d $|x|=|2x+3|$

14 (Use an x-axis from −5 to 6 and a y-axis from −1 to 11.)

For this question draw the graphs for parts **a**, **b** and **c** on the same set of axes.

a Draw the graph of $y=|x+2|$.

b Draw the graph of $y=|x-3|$.

c Hence draw the graph of $y=|x+2|+|x-3|$.

d Hence determine the values of x for which $|x+2|+|x-3|\le 9$.

Use any appropriate method, other than simply using the solve facility of a calculator, to determine the values of x for which:

15 $|x+6|=1$

16 $|x-3|=-5$

17 $|x-10|=|x-6|$

18 $|x+5|=|2x-14|$

19 $|x-3|=2x$

20 $|x+5|+|x-1|=7$

21 $|x+5|+|x-3|=8$

22 $|x-8|=|2-x|-6$

23 $|x-3|\ge|x+5|$

24 $|2x-5|\ge -5$

25 $|x-11|\ge|x+5|$

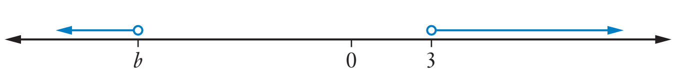
26 $|x+4|>x+2$

27 The diagram on the right shows all values of x for which

$$|2x+5| * a$$

where '$*$' is one of $<$, $\le$, $>$, or $\ge$, and a and b are constants.

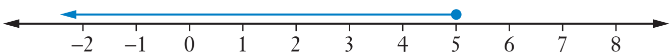
Determine which of these symbols $*$ represents and find the values of a and b.



 ISBN 9780170395274

28 The diagram on the right shows all values of x for which

$$|x-3| * |x-a|$$

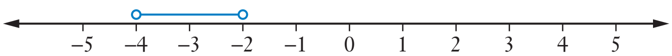


where '*' is one of <, ≤, >, or ≥, and a is a constant.

Determine which of these symbols * represents and find the value of a.

29 The diagram on the right shows all values of x for which

$$|2x+5| * |x+a|$$

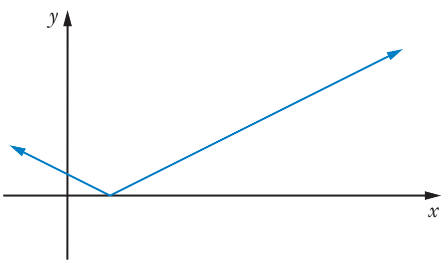


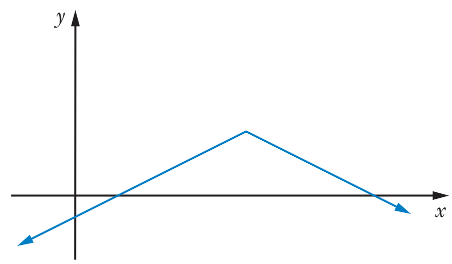
where '*' is one of <, ≤, >, or ≥, and a is a constant.

Determine which of these symbols * represents and find the value of a.

30 The graph below left shows the function $y = |0.5x - 1|$.

The graph below right shows the function $y = a|x-b| + c$, for constant a, b and c.



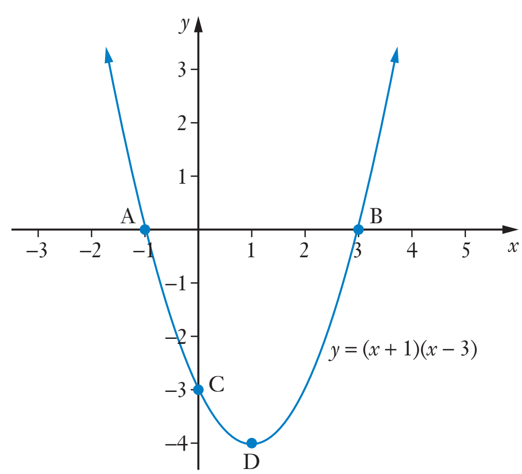


Given that the two graphs, if drawn on the same axes, would coincide for $2 \le x \le 8$, and nowhere else, determine a, b and c.

The graph of $y = \dfrac{1}{f(x)}$

The graph on the right shows the quadratic function

$$\begin{aligned} y &= x^2 - 2x - 3 \\ &= (x+1)(x-3). \end{aligned}$$



The points marked A and B are the x-axis intercepts, point C is the y-axis intercept and point D is the minimum point of the function.

- Write down the coordinates of the points A, B, C and D.
- Before turning to the next page, try to sketch the graph of

$$y = \frac{1}{(x+1)(x-3)}$$

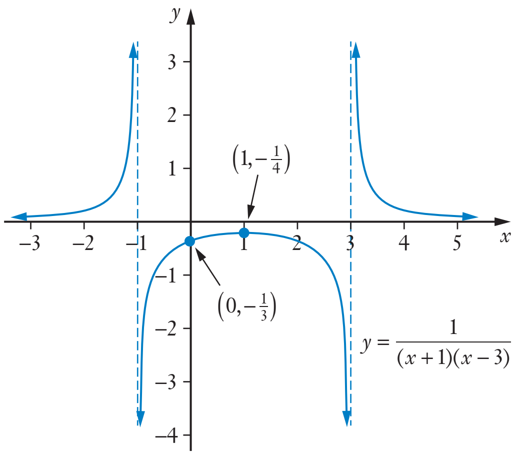
The equation and graph of $y = f(x)$ can tell us a lot about the graph of $y = \dfrac{1}{f(x)}$.

- Values of x for which $f(x) = 0$ will be values for which $\dfrac{1}{f(x)}$ will be undefined.
- Values of x for which $f(x)$ is positive (or negative) also give positive (or negative) values for $\dfrac{1}{f(x)}$.
- The x-coordinate of any local minimum (or local maximum) point on $y = f(x)$ will be the x-coordinate of a local maximum (or local minimum) point on $y = \dfrac{1}{f(x)}$.

How does your sketch of

$$y = \frac{1}{(x+1)(x-3)}$$

compare with the graph on the right?



With the quadratic $(x + 1)(x - 3)$
equalling zero when $\quad x = -1$
and when $\quad x = 3$
it follows that

$$y = \frac{1}{(x+1)(x-3)}$$

will be undefined for $\quad x = -1$
and for $\quad x = 3$,
as shown in the graph.

We say that $y = \dfrac{1}{(x+1)(x-3)}$ is **not continuous** at $x = -1$ and at $x = 3$.

Notice from the graph that as we travel along the curve approaching $x = -1$ from the left hand side the function heads off towards very large positive numbers.

We say that as x tends towards -1 from the left, y tends to positive infinity.

We write: $\quad$ As $x \to -1^-$ $\quad$ then $\quad y \to +\infty$.

If instead we approach $x = -1$ from the right hand side, the function heads off towards very large negative numbers.

I.e. $\quad$ As $x \to -1^+$ $\quad$ then $\quad y \to -\infty$.

Similarly $\quad$ As $x \to 3^-$ $\quad$ then $\quad y \to -\infty$
and $\quad$ as $x \to 3^+$ $\quad$ then $\quad y \to +\infty$.

 ISBN 9780170395274

With access to a graphic calculator we can very quickly display the graph of a function. However there are often features of the graph that should be obvious to us as soon as we see the equation of the function.

For example, we should not need to view the graph of $y = \dfrac{1}{x-1}$ to realise that:

- The function is undefined for $x = 1$. ($x = 1$ will be a vertical asymptote.)

  Indeed, as x approaches 1 from 'the greater than 1 side' (imagine substituting $x = 1.01$, $x = 1.001$, $x = 1.0001$ etc) then $y \to +\infty$.

  $$\text{As } x \to 1^+ \text{ then } y \to +\infty.$$

  and, as x approaches 1 from 'the less than 1 side' (imagine substituting $x = 0.99$, $x = 0.999$, $x = 0.9999$ etc) then $y \to -\infty$.

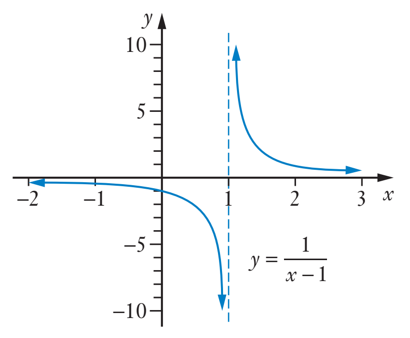
  $$\text{As } x \to 1^- \text{ then } y \to -\infty.$$

- There is no x value that gives $y = 0$ but, as $x \to +\infty$ then $y \to 0^+$
  and as $x \to -\infty$ then $y \to 0^-$

These facts can be confirmed on the graph of

$$y = \frac{1}{x-1}$$

shown on the right.



The graph of $y = \dfrac{f(x)}{g(x)}$ for $f(x)$ and $g(x)$ polynomials

The examples that follow involve sketching graphs of rational functions where both numerator and denominator are polynomials. Such functions are usually sketched by first considering some or all of

- intercepts with the axes,
- behaviour as $x \to \pm\infty$.
- vertical asymptotes,
- turning points,

and any other relevant information apparent from the equation, for example symmetry.

Also, if the order of $f(x) \geq$ the order of $g(x)$, i.e. if the algebraic fraction is improper, we may also consider rearranging the expression to 'break down' the improper fraction.

ISBN 9780170395274

Note
- The behaviour of $\frac{f(x)}{g(x)}$ as $x \to \pm\infty$ can be determined by considering the 'dominant powers' of $f(x)$ and $g(x)$.

  For example, consider $y = \frac{3x^2 + 2x - 5}{2x^2 - x + 6}$. As $x \to \pm\infty$,

  $$\begin{aligned} y &\to \frac{3x^2}{2x^2} \\ &= \frac{3}{2}. \qquad (x \neq 0) \end{aligned}$$

  The reader should check the correctness of this by viewing the graph of the function on a calculator.

- Using calculus to determine any turning points on the graph of $y = \frac{f(x)}{g(x)}$ will generally require use of the quotient rule for differentiation. It is assumed that you will have recently encountered this rule in your concurrent study of Unit Three of *Mathematics Methods*.

EXAMPLE 18

Make a sketch of the function with equation $y = \frac{x-4}{x-2}$.

Solution

The given fraction is improper so we could consider breaking it down:

$$\begin{aligned} y = \frac{x-4}{x-2} &= \frac{x-2-2}{x-2} \\ &= 1 - \frac{2}{x-2} \end{aligned}$$

If $x = 4$, $y = 0$. Cuts x-axis at (4, 0).

If $x = 0$, $y = 2$. Cuts y-axis at (0, 2).

As $x \to \pm\infty$, $y \to \frac{x}{x}$, i.e. 1.

[Indeed, from $y = 1 - \frac{2}{x-2}$, as $x \to +\infty$, $y \to 1^-$, and, as $x \to -\infty$, $y \to 1^+$.]

If $x = 2$, y is undefined. $x = 2$ is a vertical asymptote.

[Indeed, as $x \to 2^+$, $y \to -\infty$, and as $x \to 2^-$, $y \to +\infty$.]

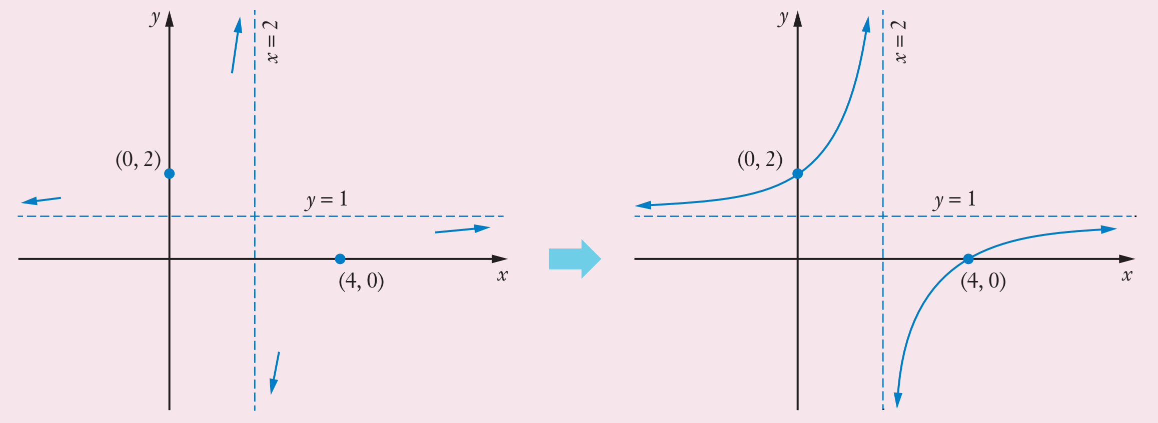
If $y = \frac{x-4}{x-2}$, using the quotient rule

$$\begin{aligned} \frac{dy}{dx} &= \frac{(x-2)(1)-(x-4)(1)}{(x-2)^2} \\ &= \frac{2}{(x-2)^2} \neq 0 \end{aligned}$$

There are no stationary points (and the gradient is always positive).

With this information (and even without the information bracketed above) a sketch can be completed. See the next page.

 ISBN 9780170395274



EXAMPLE 19

Make a sketch of the function with equation $y = \dfrac{2x - 1}{x(x+4)}$.

Solution

If $x = 0.5$, $y = 0$. Cuts x-axis at $(0.5, 0)$.

If $x = 0$, y is undefined. $x = 0$ is a vertical asymptote.
Indeed, as $x \to 0^+$, (consider, for example, $x = 0.01$) $y \to -\infty$,
and as $x \to 0^-$, (consider, for example, $x = -0.01$) $y \to +\infty$.

If $x = -4$, y is undefined. $x = -4$ is a vertical asymptote.
Indeed, as $x \to -4^+$, (consider, for example, $x = -3.9$) $y \to +\infty$,
and as $x \to -4^-$, (consider, for example, $x = -4.1$) $y \to -\infty$.

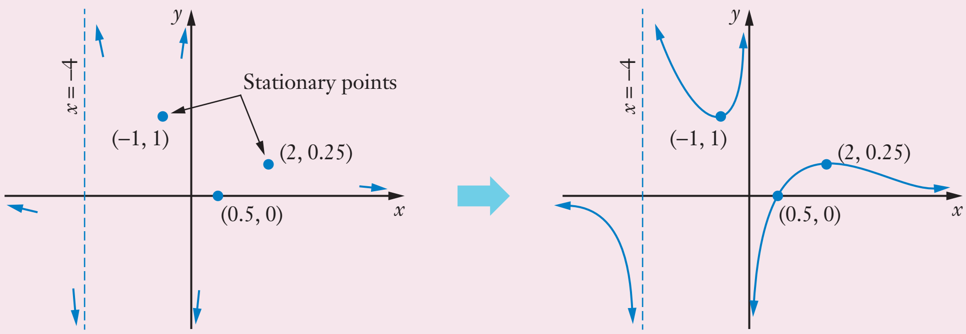
As $x \to \pm\infty$, $y \to \dfrac{2x}{x^2}$, i.e. $\dfrac{2}{x}$. As $x \to +\infty, y \to 0^+$. As $x \to -\infty, y \to 0^-$.

If $y = \dfrac{2x-1}{x(x+4)}$, using the quotient rule, and simplifying, gives $\dfrac{dy}{dx} = \dfrac{-2(x-2)(x+1)}{x^2(x+4)}$

There are stationary points when $x = -1$ and when $x = 2$.

When $x = -1, y = 1$. When $x = 2, y = 0.25$. Stationary points exist at $(-1, 1)$ and $(2, 0.25)$.

The nature of each of these can be determined by considering the gradients either side of $x = -1$ and $x = 2$, or the sketch can be completed using the facts already determined.



Note: From the shape of the graph there must also be a point of inflection (not horizontal) to the right of the local maximum.

ISBN 9780170395274

EXAMPLE 20

Sketch the graph of the function $y = \dfrac{x^2 - 4}{x - 1}$.

Solution

Note that
$$\begin{aligned} y &= \frac{x^2-4}{x-1} \\ &= \frac{(x+2)(x-2)}{x-1} \end{aligned}$$
and
$$\begin{aligned} y &= \frac{x^2-4}{x-1} \\ &= \frac{x(x-1)+x-4}{x-1} \\ &= \frac{x(x-1)}{x-1} + \frac{(x-1)-3}{x-1} \\ &= x+1-\frac{3}{x-1} \end{aligned}$$

If $x = 2$, $y = 0$. Cuts x-axis at (2, 0).

If $x = -2$, $y = 0$. Cuts x-axis at (–2, 0).

If $x = 0$, $y = 4$. Cuts y-axis at (0, 4).

As $x \to \pm\infty$, $y \to x + 1$. The line $y = x + 1$ will be an **oblique asymptote**.

As $x \to +\infty$, $y \to (x+1)^-$. As $x \to -\infty$, $y \to (x+1)^+$.

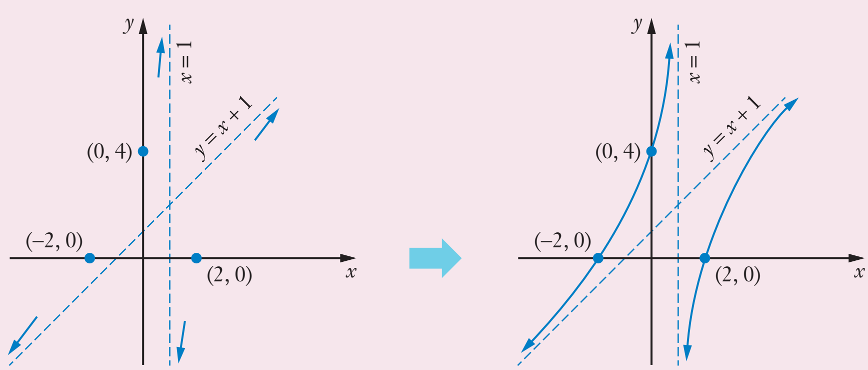
If $x = 1$, y is undefined. $x = 1$ is a vertical asymptote.

As $x \to 1^+$, $y \to -\infty$. As $x \to 1^-$, $y \to +\infty$.

If $y = \dfrac{x^2-4}{x-1}$, using the quotient rule
$$\begin{aligned} \frac{dy}{dx} &= \frac{(x-1)(2x)-(x^2-4)(1)}{(x-1)^2} \\ &= \frac{x^2-2x+4}{(x-1)^2} \end{aligned}$$
$x^2 - 2x + 4 = 0$ has no real solutions.

Hence no stationary points.

Placing this information on a graph, below left, the sketch can be completed, below right.



 ISBN 9780170395274

EXAMPLE 21

Sketch the graph of the function $y = \dfrac{x+3}{x+3}$.

Solution

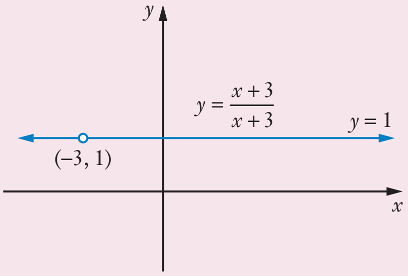
The immediate reaction may be to simply 'cancel the $(x + 3)$s', to give $y = 1$.

However, if we do this, we must remember that the initial function is undefined for $x = -3$.

Thus
$$\begin{aligned} y &= \frac{x+3}{x+3} \\ &= 1, \qquad x \neq -3 \end{aligned}$$

The function is sketched on the right.

The open circle shows a point of discontinuity.
The function is not defined for $x = -3$.



EXAMPLE 22

Sketch the graph of the function $y = \dfrac{x^2 - 4}{x(x-2)}$.

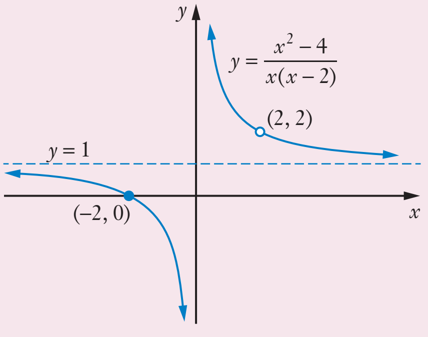
Solution

Note that
$$\begin{aligned} y &= \frac{x^2-4}{x(x-2)} \\ &= \frac{(x-2)(x+2)}{x(x-2)} \\ &= \frac{(x+2)}{x} \qquad x \neq 2. \\ &= 1 + \frac{2}{x} \qquad x \neq 2 \end{aligned}$$

The graph will be that of $y = \dfrac{1}{x}$ dilated $\updownarrow \times 2$, translated 1 unit $\uparrow$, with $x \neq 2$.

Also note that if $x = -2$, $y = 0$.

Cuts x-axis at $(-2, 0)$.



ISBN 9780170395274

EXAMPLE 23

Make a sketch of the function $y = \dfrac{x^2 + 2x + 1}{x - 2}$.

Solution

Note that
$$y = \frac{x^2 + 2x + 1}{x - 2} = \frac{(x+1)^2}{x - 2}$$
and
$$y = \frac{x^2 + 2x + 1}{x - 2} = \frac{x(x-2) + 4(x-2) + 9}{x - 2} = x + 4 + \frac{9}{x - 2}$$

If $x = -1$, $y = 0$. Cuts (or perhaps *touches*) x-axis at $(-1, 0)$.

If $x = 0$, $y = -0.5$. Cuts y-axis at $(0, -0.5)$.

As $x \to \pm\infty$, $y \to x + 4$. The line $y = x + 4$ will be an **oblique asymptote**.

As $x \to +\infty$, $y \to (x+4)^+$. As $x \to -\infty$, $y \to (x+4)^-$.

If $x = 2$, y is undefined. $x = 2$ is a vertical asymptote.

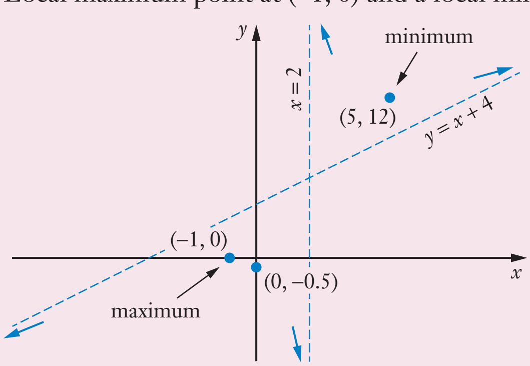
As $x \to 2^+$, $y \to +\infty$. As $x \to 2^-$, $y \to -\infty$.

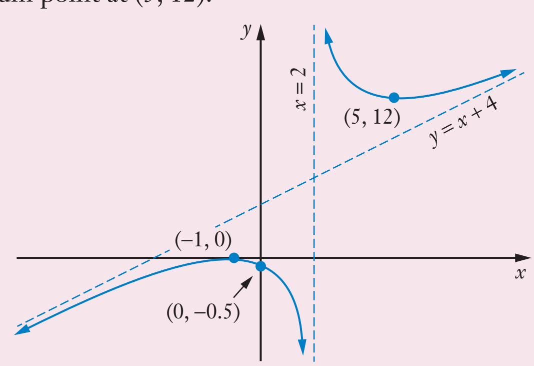
If $y = \dfrac{x^2 + 2x + 1}{x - 2}$, using the quotient rule
$$\frac{dy}{dx} = \frac{(x-2)(2x+2) - (x^2 + 2x + 1)(1)}{(x-2)^2} = \frac{x^2 - 4x - 5}{(x-2)^2} = \frac{(x-5)(x+1)}{(x-2)^2}$$

There are stationary points at $(-1, 0)$ and at $(5, 12)$. Their nature can be determined as shown below (or the sketch can be completed from the facts already determined).

	Gradient either side of $x = -1$			Gradient either side of $x = 5$		
	$x = -1.1$	$x = -1$	$x = -0.9$	$x = 4.9$	$x = 5$	$x = 5.1$
$x - 5$	–ve	–ve	–ve	–ve	zero	+ve
$x + 1$	–ve	zero	+ve	+ve	+ve	+ve
$\dfrac{(x-5)(x+1)}{(x-2)^2}$	+ve	zero	–ve	–ve	zero	+ve
	/	—	\	\	_	/

Local maximum point at $(-1, 0)$ and a local minimum point at $(5, 12)$.





 ISBN 9780170395274

Exercise 3D

For what values of x will the following functions have vertical asymptotes?

1 $y = \frac{2}{x}$ **2** $y = \frac{5}{x-1}$ **3** $y = \frac{5}{(x-3)(2x-1)}$ **4** $y = \frac{x+3}{x-3}$

What values of y cannot be obtained from each of the following functions if x can take all real values for which the function is defined?

5 $y = \frac{3}{x}$ **6** $y = 2 + \frac{3}{x}$ **7** $y = \frac{1}{x+1}$ **8** $y = \frac{x-1}{x+1}$

For each of the following, complete statements of the form:

As $x \to +\infty$ then $y \to$????

and as $x \to -\infty$ then $y \to$????.

9 $y = \frac{1}{x-5}$ **10** $y = \frac{x+2}{x-2}$ **11** $y = \frac{5x^2+7x-3}{x^2+6}$ **12** $y = \frac{3x(x+2)}{x^2+1}$

13 For $y = \frac{1}{x-3}$ copy and complete:

As $x \to 3^+$ then $y \to$???? and as $x \to 3^-$ then $y \to$????

14 For $y = \frac{1}{1-x}$ copy and complete:

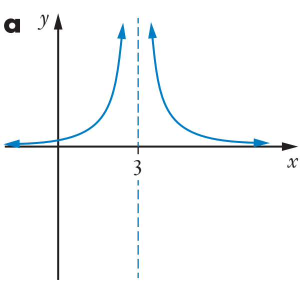
As $x \to 1^+$ then $y \to$???? and as $x \to 1^-$ then $y \to$????

15 For $y = \frac{x^5+1}{x^2}$ copy and complete:

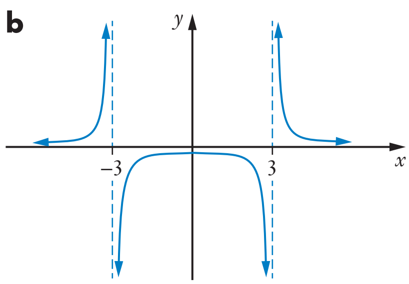
As $x \to 0^+$ then $y \to$???? and as $x \to 0^-$ then $y \to$????

16 Given that the equations of each of the following sketch graphs are in the equations box below, select the appropriate equation.

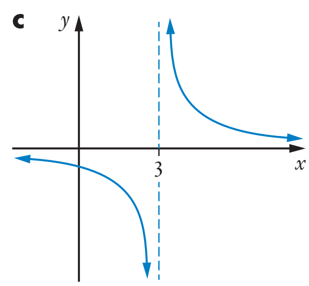
a



b



c



Equations Box

$y = \frac{1}{(3+x)^2}$ $y = \frac{1}{(x+3)(x-3)}$ $y = \frac{1}{(3-x)}$ $y = \frac{1}{(3+x)(3-x)}$ $y = \frac{1}{(x-3)^2}$ $y = \frac{1}{(x-3)}$

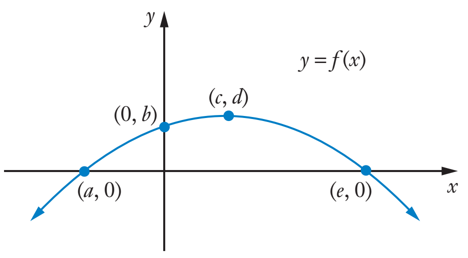
ISBN 9780170395274

17 Sketch both $y = (x+2)(x-2)$ and $y = \dfrac{1}{(x+2)(x-2)}$ on the same set of axes.

Once completed, check the correctness of your sketch by viewing the graphs of the functions on a graphic calculator.

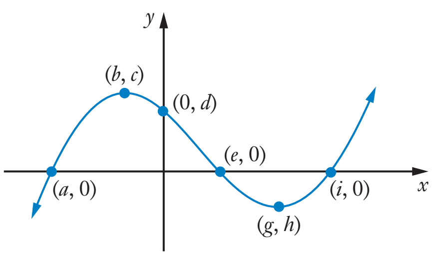
18 The graph of $y = f(x)$ is shown on the right.

Produce a sketch of $y = \dfrac{1}{f(x)}$.



19 The graph of $y = g(x)$ is shown on the right.

Produce a sketch of $y = \dfrac{1}{g(x)}$.



Without the assistance of a graphic calculator, make sketches of the graphs of the following functions.

20 $y = \dfrac{x+3}{x-1}$

21 $y = \dfrac{2x-4}{x+2}$

22 $y = \dfrac{2(x-4)}{x-4}$

23 $y = \dfrac{x^2-9}{x(x+3)}$

24 $y = \dfrac{36(2-x)}{x(x+6)}$

25 $y = \dfrac{1-x}{x(x+3)}$

26 $y = \dfrac{x}{x^2-1}$

27 $y = \dfrac{(x-4)(x-1)}{x-2}$

28 $y = \dfrac{x^2+3x}{x-1}$

29 $y = \dfrac{x^2-3x-4}{x^3-2x^2-3x}$

30 $y = \dfrac{3}{x^3-3x^2+3x}$

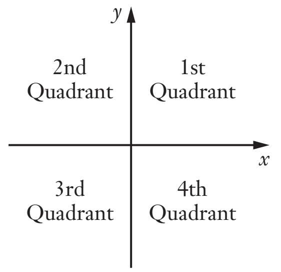
31 $y = \dfrac{x^3}{x-2}$

ISBN 9780170395274

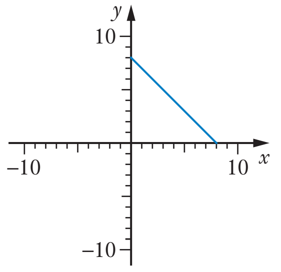
Absolute value – extension activity

How would you construct the graph of $|x| + |y| = 8$?

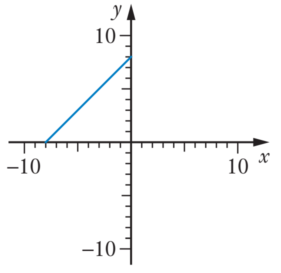
One approach is to consider each of the four quadrants in turn:



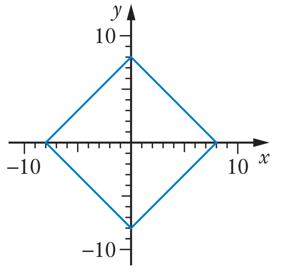
In the first quadrant $x > 0$ and $y > 0$ so we draw $x + y = 8$ in this quadrant:



In the second quadrant $x < 0$ and $y > 0$ so we draw $-x + y = 8$ in this quadrant:



Continuing in this way we would draw $-x - y = 8$ in the third quadrant and $x - y = 8$ in the fourth quadrant, to give the graph of $|x| + |y| = 8$.



Alternatively we could consider $|x| + |y| = 8$ as $y = \begin{cases} 8 - |x| & \text{for} \quad y \geq 0 \\ 8 + |x| & \text{for} \quad y < 0 \end{cases}$.

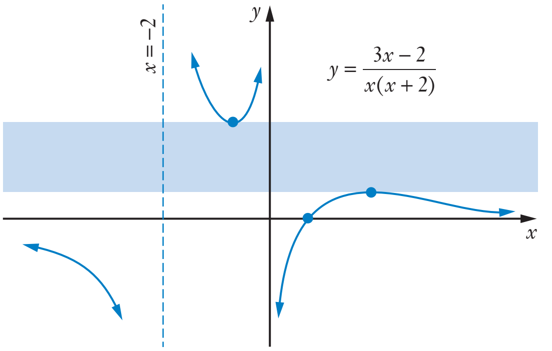
Produce graphs of each of the following:

- $|x| - |y| = 8$
- $3|x| - 2|y| = 24$
- $|x|^2 + |y| = 4|x|$
- $|y| = |x|^2 + 2|x| - 8$
- $2|x| + |y| = 10$
- $|y| = 4 - |x|^2$
- $|x|^2 + |y| = 4x$
- $|y| = |x|^2 + 2x - 8$

ISBN 9780170395274

Rational functions – extension activity

The graph of $f(x) = \dfrac{3x-2}{x(x+2)}$ is shown on the right.



Notice that there seems to be a horizontal band, shown shaded in the diagram, in which the function does not appear.

Using calculus, or a graphic calculator, to determine the coordinates of the turning points we can say that the range of $f(x)$ includes $y \le 0.5$ and also $y \ge 4.5$, but excludes $0.5 < y < 4.5$.

An alternative approach for determining this range of values y can take uses a quadratic equation approach and is shown below.

Let $$y = \frac{3x-2}{x(x+2)}$$

Then for $x \neq 0$ and $x \neq -2$

$$yx(x+2) = 3x - 2$$
$$yx^2 + 2yx = 3x - 2$$
$$yx^2 + (2y-3)x + 2 = 0 \qquad [1]$$

If $y = 0$ equation [1] reduces to $-3x + 2 = 0$, giving $x = \frac{2}{3}$, the x-axis intercept.

If $y \neq 0$ equation [1] is a quadratic equation.

$ax^2 + bx + c$ has real solutions if $b^2 - 4ac \ge 0.$

Thus from [1], for real x,

$$(2y-3)^2 - 4(y)(2) \ge 0$$
$$4y^2 - 20y + 9 \ge 0$$
$$(2y-9)(2y-1) \ge 0$$

	$y < 0.5$	$0.5 < y < 4.5$	$y > 4.5$
$2y - 9$	–ve	–ve	+ve
$2y - 1$	–ve	+ve	+ve
$(2y-9)(2y-1)$	+ve	–ve	+ve

Thus for real x we must have $y \le 0.5$ or $y \ge 4.5$ which agrees with the values mentioned earlier, found using calculus or a graphic calculator.

Before the availability of graphic calculators, the above method was used to give useful information about the range of a function. Whilst it is not suggested we would use it very often now when we can view the graph on a calculator so easily, do attempt the following by the above method and then check that your answers agree with the display from a graphic calculator.

1 $y = \dfrac{9(2x-1)}{x(x+12)}$

2 $y = \dfrac{2x+2}{x^2+x+1}$

3 $y = \dfrac{8x+7}{2(x+1)(x+2)}$

4 $y = \dfrac{x-4}{(x-3)(x-5)}$

5 $y = \dfrac{3x-10}{(x-2)(x-3)}$

6 $y = \dfrac{3x+2}{(7x^2+5x+1)}$

ISBN 9780170395274

Miscellaneous exercise three

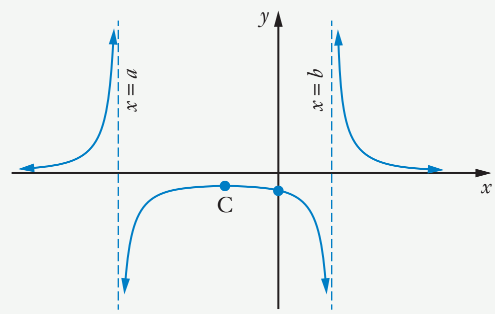
This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the Preliminary work section at the beginning of the book.

1 Without the assistance of your calculator find all values of x, real and complex, for which:

$$x^3 + 7x^2 + 19x + 13 = 0$$

2 The graph on the right shows the function

$$y = \frac{1}{(x-1)(x+3)}.$$



a State the values of a and b.

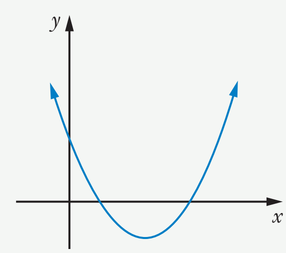
b By considering the coordinates of the turning point of

$$y = (x-1)(x+3)$$

find the coordinates of point C, the local maximum point.

3 The graph on the right shows some function $y = f(x)$.

Produce sketches of the graphs of each of the following:



a $y = -f(x)$ **b** $y = f(-x)$

c $y = |f(x)|$ **d** $y = f(|x|)$

4 Sketch the graph of

a $y = |x - a|$, $a > 0$,

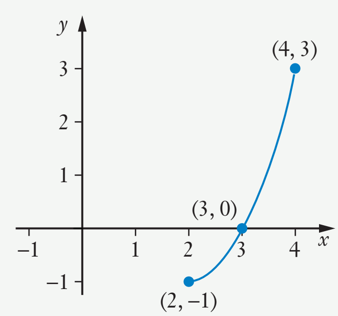
b $y = |2x - a|$, $a > 0$.

Hence determine the values of x for which $|2x - a| \le |x - a|$.

5 The graph on the right shows the function

$$f(x) = (x-2)^2 - 1$$

for a restricted domain.



a State the domain and range of $f(x)$ shown in the diagram.

With the domain of $f(x)$ as shown in the graph:

b state the domain and range of $f^{-1}(x)$,

c show $f(x)$ and $f^{-1}(x)$ on a single sketch,

d find an algebraic expression for $f^{-1}(x)$.

ISBN 9780170395274

6 Find the values of the scalars p and q in each of the following cases given that **a** and **b** are non-parallel, non-zero vectors.

a $p\mathbf{a} = q\mathbf{b}$

b $(p - 3)\mathbf{a} = q\mathbf{b}$

c $(p + 2)\mathbf{a} = (q - 1)\mathbf{b}$

d $p\mathbf{a} + 2\mathbf{b} = 3\mathbf{a} - q\mathbf{b}$

e $p\mathbf{a} + q\mathbf{a} + p\mathbf{b} - 2q\mathbf{b} = 3\mathbf{a} + 6\mathbf{b}$

f $p\mathbf{a} + 2\mathbf{a} - 2p\mathbf{b} = \mathbf{b} + 5q\mathbf{b} - q\mathbf{a}$

7 If $\mathbf{a} = 2\mathbf{i} + 4\mathbf{j}$ and $\mathbf{b} = 5\mathbf{i} - 3\mathbf{j}$ express each of the following in the form $p\mathbf{a} + q\mathbf{b}$, where p and q are suitably chosen scalars.

a $-9\mathbf{i} + 21\mathbf{j}$

b $4\mathbf{i} - 18\mathbf{j}$

c $-7\mathbf{i} + 12\mathbf{j}$

d $-34\mathbf{i} + 23\mathbf{j}$

8 **a** Determine, in exact cartesian form, the complex number z for which:

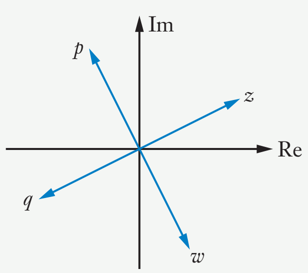
$$\frac{1}{z} = \frac{-3 + 2\sqrt{3}i}{3 + 5\sqrt{3}i}$$

b Write the z of part **a** in exact polar form.

9 Simplify $\dfrac{1}{4\,\text{cis}\left(-\dfrac{\pi}{6}\right)}$ giving an exact answer in the form $a + ib$.

10 The four complex numbers z, p, q and w shown on the Argand diagram on the right all have the same magnitude and p is perpendicular to z, q is perpendicular to p, and w is perpendicular to q.

Express p, q and w in terms of z.



11 If $z = 2\,\text{cis}\left(\dfrac{\pi}{4}\right)$ and $w = 1\,\text{cis}\left(\dfrac{\pi}{6}\right)$ express each of the following in the form $r\,\text{cis}\,\theta$ with $r \geq 0$ and $-\pi < \theta \leq \pi$.

a zw

b $\dfrac{z}{w}$

c w^2

d z^3

e w^9

f z^9

12 Express $(-\sqrt{3} + i)$ in the form $r\,\text{cis}\,\theta$ with $r \geq 0$ and $-\pi < \theta \leq \pi$.
Hence use de Moivre's theorem to simplify $(-\sqrt{3} + i)^{12}$.

ISBN 9780170395274